Local Geometry of Self-similar Sets: Typical Balls, Tangent Measures and Asymptotic Spectra.

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Abstract

We analyse the local geometric structure of self-similar sets with open set condition through the study of the properties of a distinguished family of spherical neighbourhoods, the typical balls. We quantify the complexity of the local geometry of self-similar sets, showing that there are uncountably many classes of spherical neighbourhoods that are not equivalent under similitudes. We show that, at a tangent level, the uniformity of the Euclidean space is recuperated in the sense that any typical ball is a tangent measure of the measure ν at ν-a.e. point, where ν is any self-similar measure. We characterise the spectrum of asymptotic densities of metric measures in terms of the packing and centred Hausdorff measures. As an example, we compute the spectrum of asymptotic densities of the Sierpinski gasket.

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1 Introduction and main results

In order to gauge the vastness of the set of spherical neighbourhoods of a metric space X, it is useful to consider the quotient spaces $Sph_X/\simeq_{\mathcal{F}}$, where Sph_X is the set of spherical neighbourhoods of X and $\simeq_{\mathcal{F}}$ is the equivalence class associated with some group $\mathcal{F}$ of self-mappings of X : $B \simeq_{\mathcal{F}} B' \Leftrightarrow B = f(B')$ for B, $B' \in Sph_X$ and some $f \in \mathcal{F}$. The regularity of the Euclidean space $\mathbb{R}^n$ is made clear by the fact that if $\mathcal{S}_n$ is the set of similarities of $\mathbb{R}^n$, then $Sph_{\mathbb{R}^n}/\simeq_{\mathcal{S}_n}$ consists of a unique equivalence class.

In this paper, we study the local geometry of a self-similar set $E \subset \mathbb{R}^n$ satisfying the open set condition (OSC), geometry which is described by the spherical neighbourhoods of E as a metric subspace of $\mathbb{R}^n$, i.e. by restricted balls of the form $B \cap E$, where B is a Euclidean ball. For general points x, $y \in E$, if $B(x,d)$ denotes the closed Euclidean ball centred at x and with radius d, then $B(x,d) \cap E$ and $B(y,d) \cap E$ are not equivalent by translation, and $B(x,d) \cap E$ and $B(x,d') \cap E$ with $d \neq d'$ are not homothetic-equivalent. Using classical tools of fractal geometry, namely, the s-densities of metric measures on balls (see Definitions 21 and 22), and results by Marstrand [1], Hutchinson [2] and Preiss [3], together with the results in Sec. 3.2, we are able to prove that, for general self-similar sets with OSC, there are uncountably many equivalence classes in the quotient spaces $Sph_E/\simeq_{\mathcal{S}_n}$. This gives account of the complexity of the purely deterministic self-similar geometry.

In spite of these facts, the literature has established the existence of a strong kind of regularity, on a tangent level and on average, in the neighbourhoods of a self-similar set.

Recall that a *self-similar set* is defined as the unique compact set $E \subset \mathbb{R}^n$ that satisfies the basic equation of self-similarity

$$E = \cup_{i=0}^{m-1} f_i(E). \tag{1}$$

for a given system $\Psi = \{f_i\}_{i \in M}$, $M := \{0, 1, \ldots, m-1\}$ of contractive similitudes in $\mathbb{R}^n$. We shall assume that the system Ψ satisfies the OSC, meaning that there is an open set $\mathcal{O} \subset \mathbb{R}^n$ such that $f_i(\mathcal{O}) \subset \mathcal{O}$ for all $i \in M$ and $f_i(\mathcal{O}) \cap f_j(\mathcal{O}) = \varnothing$ for i, $j \in M$, $i \neq j$. We shall refer to such a set

$\mathcal{O}$ as a *feasible open set* for Ψ. We can assume, without loss of generality, as we shall from now on, that $\mathcal{O}\cap E \neq \varnothing$ holds, also called *strong open set condition* (SOSC) (cf. [4] and [5], see also [6]). If $f_i(E)\cap f_j(E)=\varnothing$ for $i,\ j\in M,\ i\neq j$, it is said that the *strong separation condition* (SSC) holds, in which case the OSC is also fulfilled.

We want to understand the local geometry of E through the study of the local behaviour of the metric s-measures,

$$\mathcal{M}^s\lfloor_E := \left\{\mu,\ \mathcal{H}^s\lfloor_E,\ \mathcal{H}^s_{Sph}\lfloor_E,\ C^s\lfloor_E,\ P^s\lfloor_E\right\} \tag{2}$$

where s is the *similarity dimension* of E, $\dim E$, that is, the unique real number s that satisfies $\sum_{i\in M} r_i^s = 1$, r_i being the contraction constant of the similarity $f_i,\ i\in M$. Here $\beta\lfloor_E$ stands for a measure β restricted to the set E. The measures

$$\mathcal{M}^s := \left\{\mathcal{H}^s,\ \mathcal{H}^s_{Sph},\ C^s,\ P^s\right\} \tag{3}$$

are the s-dimensional Hausdorff measure, spherical Hausdorff measure, centred Hausdorff measure and packing measure, respectively. Any two measures in $\mathcal{M}^s\lfloor_E$ are multiple of each other, moreover, in the case that s takes the integer value n, they are also multiple of the n-dimensional Lebesgue measure. Each measure in $\mathcal{M}^s\lfloor_E$ highlights different basic geometric properties of subsets of $\mathbb{R}^n$. For $\alpha\in\mathcal{M}^s\lfloor_E$, $0<\alpha(E)<\infty$ holds and E is called an s-set (see [7] for further details and Sec. 2.2 for the definitions of the measures in $\mathcal{M}^s$). We shall present in Sec. 2.1 below the *natural probability measure* μ. For the time being, we can see it as the normalised measure, $\frac{\alpha}{\alpha(E)}$ of any other $\alpha\in\mathcal{M}^s\lfloor_E$.

The results in this paper about the regularity of the metric measures are also shared by the wider class of self-similar measures, $\mathcal{M}_{\mathcal{S}}(E)$ (see [2] and Sec. 2.1 for a definition). Whereas the metric measures, $\mathcal{M}^s$, convey a strong geometric meaning, self-similar measures are an essential tool in multifractal analysis of logarithmic densities, a topic that has generated a vast amount of literature for the past 30 years.

1.1 Scenery flow, tangent distribution and tangent measures

Let ν be a Radon measure on $\mathbb{R}^n$ and let x be a point in the support of ν. We can access the local geometry of $\nu\lfloor_E$ around x through the following zooming process: let $T_{x,t}(y)=t(y-x),\ t>0$, be the homothety that maps the ball $B(x,t^{-1})$ onto the unit ball $D:=B(0,1)$. Let $\nu_{x,t}$ be the probability measure on D obtained from the normalisation of the restriction to D of the image measure of $\nu\lfloor_E$ under the homothety $T_{x,t}$. If $\mathcal{M}(D)$ denotes the set of Radon measures on D, then the mapping $t \to \nu_{x,t}$ can be considered as a measure-valued time series that takes values in the metric space $\mathcal{M}(D)$ endowed with the weak topology. This time series is called *scenery flow* of ν around x (cf. [8]). The empirical distributions $\Phi_{x,t}(\nu)$, $t > 0$, associated with such "time" series, are probability measures on $\mathcal{M}(D)$ (so they belong to the set $\mathcal{M}(\mathcal{M}(D))$ of Radon measures on $\mathcal{M}(D)$). The empirical distribution $\Phi_{x,t}(\nu)$ gives weight to a set $A \subset \mathcal{M}(D)$ according to the rate of the *time* interval $[0, t]$ that the "empirical" data $\delta_{\nu_{x,t}}$ (unit mass at $\nu_{x,t}$) stay in A. If the empirical distribution $\Phi_{x,t}(\nu)$ converges to a limit $\Phi_x(\nu)$ as t tends to infinity, then the limiting distribution $\Phi_x(\nu)$ is called the *tangent distribution* of ν at x (see [9]).

S. Graf [10] proved that if E is a self-similar set with OSC and $\nu \in \mathcal{M}_{\mathcal{S}}(E)$, then the limit $\Phi_x(\nu)$ exists ν-a.e. x, and it does not depend on x. Moreover, he constructed an explicit formula for the tangent distribution. This author gave credit for the first of these results to C. Bandt by [9], and Bandt in [11] gives credit for the same result to S. Graf by [10] (indeed a most refreshing case). M. Arbeiter [12], C. Bandt [11] and A. Pyörälä [13] extended these results in different ways. The uniqueness and independence of the limit $\Phi_x(\nu)$ from x is what M. Gavish, [14], calls, when displayed by a measure, the *uniform scaling scenery property* of such a measure. This means that, at a tangent level and in this sense, the flow scenery recovers the uniformity of the Euclidean space.

Remark 1 *There is another way to pass to the limit at the tangent level that leads to* tangent measures, *a concept prior to tangent distributions introduced by D. Preiss [3]. There, starting from a measure ν in the set $\mathcal{M}(\mathbb{R}^n)$ of Radon measures on $\mathbb{R}^n$, he considers unrestricted zoomings $\nu_{x,t}$ of ν at x by homotheties $T_{x,t}$ as above. Instead of performing an averaging procedure, Preiss considers non-null and locally finite limits, in the vague topology of $\mathcal{M}(\mathbb{R}^n)$, of sequences*

$$\{c_n \nu_{x,t_n}\} \quad \textit{with} \quad t_n \overset{n\to\infty}{\to} \infty \quad \textit{and} \quad c_n > 0.$$

Such limit points are called tangent measures of ν at x, and $Tan(\nu, x)$ denotes the set of all such limits. In our approach, following C. Bandt [9], the measures ν_{x,t_n} are restricted and normalised zoomings, but the zoomings are through general expanding similitudes, rather than only homotheties.

Let $\mathcal{I}_n$ be the group of isometries of $\mathbb{R}^n$. We may define, in the set $\mathcal{M}(\mathbb{R}^n)$, the equivalence relationship

$$\alpha \cong \beta \Leftrightarrow \text{there is a } g \in \mathcal{I}_n \text{ and a } \lambda > 0 \text{ such that } \beta = \lambda\,(g_\sharp(\alpha)), \tag{4}$$

where $g_\sharp(\alpha)$ is the *image measure* of α under g, i.e. $g_\sharp(\alpha)(A) = \alpha(g^{-1}(A))$ for α-measurable $A \subset \mathbb{R}^n$. Thus, we identify two measures if they are equal up to an isometry (see, for instance, [11], where equivalent measures up to isometries are identified in the construction of tangent measures), and we also identify all measures in the half-straight line $\{\lambda\alpha : \lambda > 0,\ \alpha \in \mathcal{M}(\mathbb{R}^n)\}$. For $\alpha \in \mathcal{M}(\mathbb{R}^n)$, let $\widetilde{\alpha}$ denote the equivalence class in $\mathcal{M}(\mathbb{R}^n)/\cong$ to which α belongs, i.e.

$$\widetilde{\alpha} = \{\beta \in \mathcal{M}(\mathbb{R}^n) : \beta \cong \alpha\} \tag{5}$$

Given a measure $\nu \in \mathcal{M}(\mathbb{R}^n)$, we now consider the zoomings ν_{x,t_n} be of the form $(g_n)_\sharp\ \nu\lfloor_{B(x,dt_n^{-1})}$ where g_n is a similitude of contraction ratio t_n, $d \leq 1$, and $x \in spt(\nu)$ (see (23)). We define the quotient space $\widetilde{\mathcal{M}}(\mathbb{R}^n)$ and the set of *tangent equivalence classes of measures,* $\widetilde{Tan}(\nu, x)$, by

$$\widetilde{\mathcal{M}}(\mathbb{R}^n) = \{\widetilde{\alpha} : \alpha \in \mathcal{M}(\mathbb{R}^n)\} \tag{6}$$

$$\widetilde{Tan}(\nu, x) = \left\{\widetilde{\alpha} : \text{there is a sequence } c_n\nu_{x,t_n} \xrightarrow[n\to\infty]{w} \alpha, \text{with } t_n \underset{n\to\infty}{\longrightarrow} \infty,\ \alpha \neq 0 \text{ and } \alpha \in \mathcal{M}(\mathbb{R}^n)\right\} \tag{7}$$

where $\overset{w}{\to}$ denotes the weak convergence of measures on $\mathcal{M}(\mathbb{R}^n)$. It turns out that, in the course of our research, the case in which the convergence of the magnifications occurs in the strong topology of measures in $\mathcal{M}(\mathbb{R}^n)$ is relevant (see Sec. 1.2 below for a discussion of this result). We shall write $\widetilde{Tan}^{st}(\nu, x)$ for the set of equivalence classes, w.r.t. $\cong$, of such strong limits.

Remark 2 *In our definition (7) any two zoomings, $\beta = (g_n)_\sharp\ \nu\lfloor_{B(x,dt_n^{-1})}$ and $\beta' = (h_n)_\sharp\,\nu\lfloor_{B(x,dt_n^{-1})}$ of a given spherical neighbourhood $B(x, dt_n^{-1})$ are considered as valid steps in the construction of a tangent limiting measure α, where g_n, h_n are different similitudes. This can be considered as the identification of β and β' as equivalent zoomings. Notice that $\beta' = \left(g_n^{-1} \circ h_n\right)_\sharp \beta$ and that $g_n^{-1} \circ h_n$ is an isometry. Thus, the equivalence relationship (4) and the definition in (7) are consistent.*

In contrast to the enlightening results obtained in [10], [11] and [12] on the uniform scaling scenery property of self-similar measures, to the best of our knowledge, the members of $Tan(\nu, x)$ for $\nu \in \mathcal{M}_{\mathcal{S}}(E)$ remain unknown. Several natural issues arise here: What is the relationship between $\Phi_x(\nu)$

and $Tan(\nu, x)$? What do the measures in $Tan(\nu, x)$ look like? Do they display some uniform property? As for the first question, see Proposition 1 in [15]. Below, we give a partial answer to the second and third questions for measures in $\mathcal{M}_{\mathcal{S}}(E)$ (see (10) and Theorem 12).

1.2 Typical balls

A distinguished class of neighbourhoods of E, in terms of which our results are expressed, is the class of *typical balls.*

Definition 3 *A ball $B(x,d)$ is said to be typical if $x \in E$ and $B(x,d) \subset \mathcal{O}$, where $\mathcal{O}$ is some feasible open set. We shall write $\mathcal{B}$ for the set of typical balls.*

The family of typical balls is invariant under the semigroup G generated by Ψ (see Sec. 2), since, for $f \in G$, it follows from $f(\mathcal{O}) \subset \mathcal{O}$ that $f(\mathcal{B}) \subset \mathcal{B}$ holds. Consider now the set of *typical spherical $\mathcal{B}$-measures,*

$$\mathcal{M}_{\mathcal{S}}(\mathcal{B}) := \{\alpha\lfloor B : B \in \mathcal{B},\ \alpha \in \mathcal{M}_S(E)\}\,. \tag{8}$$

It is well known [2] that, for any $x \in E$, the set $\{f(x) : f \in G\}$ is dense in E, so the balls in $\mathcal{B}$ are typical in the sense that, if $B \in \mathcal{B}$, then similar copies of B are densely spread over E at small scales by the action of G. These copies are a countable set of balls. As Theorem 12 shows, the measures in $\mathcal{M}_{\mathcal{S}}(\mathcal{B})$ are also typical in a deeper sense since, for any $f \in G$, $B \in \mathcal{B}$ and $\alpha \in \mathcal{M}_{\mathcal{S}}(\mathcal{B})$, the equality $\alpha\lfloor_{f(B)} = p_f f_\sharp(\alpha\lfloor_B)$ holds for a certain constant $p_f < 1$ associated with f. This means that the images of typical balls are identical copies, up to the constant p_f, to the original ones not only as subsets, but also from the point of view of any property expressible in terms of self-similar measures. Moreover, in Theorem 12 it is shown that, for any typical ball $B(x,d)$, for any measure $\alpha \in \mathcal{M}_{\mathcal{S}}(E)$ and for all points y in a set $\widehat{E}$ with full α-measure, there is a sequence of balls $\{B(y,d_k)\}$ with $d_k \to 0$, a sequence $\{f_k\}$ of similitudes in G and constants $p_{f_k}^{-1} \to \infty$, such that

$$p_{f_k}^{-1}\left(f_k^{-1}\right)_\sharp \left(\alpha\lfloor_{(B(y,d_k))}\right) \xrightarrow[k\to\infty]{st} \alpha\lfloor_{B(x,d)}, \tag{9}$$

where the convergence in (9) is in the sense of the strong topology of Radon measures.

Theorem 12 also states that, for all $x \in \widehat{E}$ and $\alpha \in \mathcal{M}_{\mathcal{S}}(\mathcal{B})$,

$$\widetilde{\mathcal{M}}_{\mathcal{S}}(\mathcal{B}) \subset \widetilde{Tan}^{st}(\alpha, x) \tag{10}$$

holds, where

$$\widetilde{\mathcal{M}}_{\mathcal{S}}(\mathcal{B}) = \{\widetilde{\alpha} : \alpha \in \mathcal{M}_{\mathcal{S}}(\mathcal{B})\}\,, \tag{11}$$

(see (5) for the notation $\widetilde{\alpha}$).

The results above imply that the use of general zooming similitudes, grants the strong convergence of the zoomings to the tangent measures, whereas in the ordinary spaces of tangent measures, where only homotheties are allowed, convergence can only be ensured in a weak topology sense. See Sec. 3.1.1 below for further details on identifications and topologies of measures.

Remark 4 *Putting the results in Sec. 3.3, described in the first paragraph of this section, together with (10), we see that the self-similar scenery at $x \in E$ depends on x on large scales, meaning that there is a broad variety of balls $B(x,d)$ for varying x that, moreover, also vary with d for fixed x. Additionally, on a tangent scale, for each $\alpha \in \mathcal{M}_{\mathcal{S}}(\mathcal{B})$ and each $x \in \widehat{E}$, each typical class of measures in $\widetilde{\mathcal{M}}_{\mathcal{S}}(\mathcal{B})$ is a feasible outcome of the zooming process of α at x, so there is a wide variety of limiting measures in $\widetilde{Tan}^{st}(\alpha,x)$, $x \in \widehat{E}$. The uniformity of the self-similar setting emerges here in the fact that the inclusion $\widetilde{\mathcal{M}}_{\mathcal{S}}(\mathcal{B}) \subset \widetilde{Tan}^{st}(\alpha,x)$ stands true for any $x \in \widehat{E}$, so all the points in $\widehat{E}$ share the set $\widetilde{\mathcal{M}}_{\mathcal{S}}(\mathcal{B})$ of tangent measures.*

1.3 Spectrum of local densities of a self-similar set: the Sierpinski gasket case

The relevance of the typical balls is stressed by the connection between typical balls and the spectrum of densities, which in turn determines some basic geometric features of E.

Let $\alpha \in \mathcal{M}(\mathbb{R}^n)$, $0 \le s \le \infty$ and $x \in \mathbb{R}^n$. The *upper and lower spherical* s-densities of α at x are defined, respectively, by

$$\overline{\theta}^s_\alpha(x) = \limsup_{d\to 0}\ \theta^s_\alpha(x,d), \tag{12}$$

$$\underline{\theta}^s_\alpha(x) = \liminf_{d\to 0}\ \theta^s_\alpha(x,d), \tag{13}$$

where the s-density of the ball $B(x,d)$, $\theta^s_\alpha(x,d)$, is given by

$$\theta^s_\alpha(x,d) = \frac{\alpha(B(x,d))}{(2d)^s}.$$

Here the zooming process is summarised in only two scalars, (12) and (13). If $\overline{\theta}^s_\alpha(x) = \underline{\theta}^s_\alpha(x)$, then we write $\theta^s_\alpha(x)$ for the common value and call it s-density of α at x. Densities and their connections to their underlying measures have been studied extensively in the context of geometric measure theory. A major contribution from Marstrand (Marstrand's theorem, [1]) asserts that, in the Euclidean setting, if the s-density $\theta^s_\alpha(x)$ exists in a set with a finite and positive α-measure with $\alpha \in \mathcal{M}(\mathbb{R}^n)$, then s is an integer.

The widest class of subsets of Euclidean spaces that are s-sets (i.e. sets with a finite and positive α-measure) is either the class of self-similar sets that satisfy the OSC, with s being their *similarity dimension* (see (2)), or some variations of it, like the Mauldin and Williams graph-directed constructions, cf. [16], and controlled Moran constructions, cf. [17]. Here, we are interested in the case in which the self-similar set is unrectifiable and therefore $\underline{\theta}^s_\alpha(x)$ and $\overline{\theta}^s_\alpha(x)$ do not coincide in subsets with a positive α-measure, case that comprises self-similar sets with non-integer similarity dimension, and also the case of self-similar sets with an integer dimension and satisfying the strong separation condition (see the proof of Corollary 25). This leads to the following definition of *asymptotic spectrum* of densities of a given measure α at a point and, more in general, in a subset of points.

Definition 5 *Given a subset $A \subset \mathbb{R}^n$, we define the* asymptotic spectrum of (non-logarithmic) spherical s-densities, *$Spec(\alpha, A)$, for a locally finite measure α by*

$$Spec(\alpha, A) = \left\{ \lim_{k\to\infty} \theta^s_\alpha(x, d_k) : x \in A \text{ and } \lim_{k\to\infty} d_k = 0 \right\}. \tag{14}$$

We insert the non-logarithmic epithet above because there is a ample literature on the so-called *multifractal spectrum* of logarithmic spherical densities. This literature also focuses on the limiting behaviour of α on small balls, but the interest is in the upper and lower limits of the quotients $\frac{\log \alpha(B(x,d))}{\log d}$ when $d \to 0$ (for $x \in E$) and, in particular, in the fractal dimension of both the (α-null) sets where these limits exist and take particular values [18] and the sets of *divergence points* (see [19], [20], [21]) where the limits do not coincide. Much less is known about the behaviour of non-logarithmic densities, and the research in this paper can be considered a preliminary step in that direction.

In particular, in Sec. 3, Theorem 14, we present the knowledge to date about the spectrum of non-logarithmic α-densities, $\alpha \in \mathcal{M}^s\lfloor_E$, of self-similar sets E that satisfy the OSC. In particular, we show that $Spec(\alpha, x)$ is contained in the closed interval $\left[\frac{\alpha(E)}{P^s(E)}, \frac{\alpha(E)}{C^s(E)}\right]$ for all x in a subset $\widehat{E}$ of E with a full α-measure. There arises a natural class of self-similar sets with nice properties, the *α-exact self-similar sets* (see Definition 16), which are sets for which the endpoints of such interval belong to $Spec(\alpha, x)$, $x \in \widehat{E}$. Whereas the results for general self-similar sets with OSC presented in Sec. 3 are of a qualitative nature, in Sec. 4 we shall focus on our prime example of α-exact self-similar set, the Sierpinski gasket S, and exploit its regularity to accurately approximate the range of values taken by its spectrum, which is the content of Theorem 26. Moreover, we give a full characterisation of the spectrum of all the points in S, which is given by the union of two closed intervals of positive length, namely,

$$Spec(\alpha, S) = \left[\alpha(S)\underline{\theta}^s_\mu(z_0), \alpha(S)\overline{\theta}^s_\mu(z_0)\right] \cup \left[\frac{\alpha(S)}{P^s(S)}, \frac{\alpha(S)}{C^s(S)}\right], \ \alpha \in \mathcal{M}^s\lfloor_S,$$

where $z_0 := (0,0)$. Using the numerical approximations of $\underline{\theta}^s_\mu(z_0)$, $\overline{\theta}^s_\mu(z_0)$ obtained in Sec. 4 and of $P^s(S)$ and $C^s(S)$ obtained in [22] and [23], we can also show that these two intervals are disjointed. In the case that $\alpha \in \{\mu, P^s\lfloor_S, C^s\lfloor_S\}$, we have numerical estimations of these two disjointed intervals. The Sierpinski gasket is, as far as we know, the first connected self-similar with non-integer dimension for which the entire spectrum has been computed.

2 Notation and preliminaries

The self-similar set E given in (1) can be parametrised as $E = \{\pi(i) : i \in \Sigma\}$ with parameter space $\Sigma := M^\infty$ and *geometric projection mapping* $\pi : \Sigma \to E$ given by $\pi(i) = \cap_{k=1}^\infty f_{i(k)}E$, where $i(k)$ denotes the curtailment $i_1 \dots i_k \in M^k$ of $i = i_1 i_2 \dots \in \Sigma$ and $f_{i_1 \dots i_k} = f_{i_1} \circ f_{i_2} \circ f_{i_3} \circ \dots f_{i_k}$. We adopt the convention $M^0 = \varnothing$ and write $M^* = \cup_{k=0}^\infty M^k$ for the set of words of finite length. Expressed in this notation, the semigroup generated by Ψ can be written as $G = \{f_i : i \in M^*\}$.

For any $i \in M^*$, we denote by E_i the cylinder sets $f_i(E)$, and if $i \in M^0$, then $f_i(E) := E$. The sets E_i are called k-cylinders if $i \in M^k$. We also shorten the notation $f_i(A)$ to A_i for a general set $A \subset \mathbb{R}^n$. We write $r_i := r_{i_1} r_{i_2} \dots r_{i_k}$ for the contraction ratio of the similitude f_i.

Moreover, $\sigma : \Sigma \to \Sigma$ shall stand for the *shift* map given by $\sigma(i_1 i_2 i_3 \dots) = i_2 i_3 i_4 \dots$ The code shift can be projected (as a correspondence) onto E, yielding the *geometric shift*

$$\mathcal{T}(x) := \pi \circ \sigma \circ \pi^{-1}(x), \tag{15}$$

$x \in E$. The *shift orbit* of $x \in E$ is given by $\left\{\mathcal{T}^k(x) : k \in \mathbb{N}\right\}$.

Remark 6 *Observe that $x \in \mathcal{T}^k(A)$ if and only if $f_i(x) \in A$ for some $i \in M^k$.*

2.1 Self-similar measures

Let $\mathcal{P}(\mathbb{R}^n)$ be the space of compactly supported probability Borel measures on $\mathbb{R}^n$, let $\mathbf{p} = (p_0, ..., p_{m-1}) \in \mathbb{R}^m$ be a probability vector and let $\mathbf{M_p}$: $\mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n)$ be the Markov operator defined by

$$\mathbf{M_p}(\alpha) = \sum_{i=0}^{m-1} p_i \alpha \circ f_i^{-1},\ \alpha \in \mathcal{P}(\mathbb{R}^n).$$

The unique fixed point of the contractive operator $\mathbf{M_p}$ is called the *self-similar measure* $\mu_\mathbf{p}$; that is,

$$\mu_\mathbf{p} = \sum_{i \in M} p_i \mu_\mathbf{p} \circ f_i^{-1}. \tag{16}$$

Moreover,

$$\mathbf{M}_\mathbf{p}^k(\alpha) = \sum_{i \in M^k} p_i \alpha \circ f_i^{-1} \underset{k \to \infty}{\overset{w}{\longrightarrow}} \mu_\mathbf{p} \tag{17}$$

for any $\alpha \in \mathcal{P}(\mathbb{R}^n)$, where, for $i \in M^k$, $p_i := p_{i_1} \cdots p_{i_k}$. Here $\mathbf{M}_\mathbf{p}^k$ is the k-th iterate of $\mathbf{M_p}$ (see [2] and [24] for further details). Set

$$\mathcal{M}_\mathcal{S}(E) := \left\{\mu_\mathbf{p} : \sum_{i=0}^{m-1} p_i = 1,\ p_i > 0,\ i = 0, ..., m-1\right\}. \tag{18}$$

For $\mathbf{p}_s := (r_0^s, ..., r_{m-1}^s)$, where s is the similarity dimension of E (recall that r_i is the contraction constant of the similarity f_i, $i \in M$), the measure $\mu_{\mathbf{p}_s}$ is called the *natural probability measure* on E. Furthermore, if $\alpha \in \mathcal{M}^s\lfloor_E$ (see (2) for notation), then

$$\mu := \mu_{\mathbf{p}_s} = \frac{\alpha}{\alpha(E)} \tag{19}$$

(see [25]).

Notice that, whereas the measures in $\mathcal{M}^s$ (see (3) for notation) convey an strong geometrical meaning, the measures $\mu_\mathbf{p}$ in $\mathcal{M}_\mathcal{S}(E)$ do not. They are concentrated in dense subsets of E, $E_\mathbf{p}$, whose dimension is given by $\dim(E_\mathbf{p}) = s_\mathbf{p} := \frac{\sum_{i=0}^{m-1} p_i \log p_i}{\sum_{i=0}^{m-1} p_i \log r_i}$, but the measure $\mu_\mathbf{p}$ is singular w.r.t. the measures $\mathcal{H}^{s_\mathbf{p}}$ and $P^{s_\mathbf{p}}$, except for $\mathbf{p} = \mathbf{p}_s$, in which case all these measures are mutually multiple (see [26] and [27]).

2.2 Metric measures

We now briefly recall metric measures. They are the classical tools for analysing the geometric properties of subsets of $\mathbb{R}^n$.

The *Hausdorff centred measure*, $C^s(A)$, of a subset $A \subset \mathbb{R}^n$, was defined by Saint Raymond and Tricot [28] in a two-step process. First, the premeasure $C_0^s(A)$ is defined for any $s > 0$ by

$$C_0^s(A) = \lim_{\delta\to 0} \inf \left\{ \sum_{i=1}^{\infty} (2d_i)^s \ : \ 2d_i \leq \delta,\ i = 1, 2, \dots \right\}, \tag{20}$$

where the infimum is taken over all coverings, $\{B(x_i, d_i)\}_{i\in\mathbb{N}^+}$, of A by closed balls $B(x_i, d_i)$ centred at points $x_i \in A$. Then, the *centred Hausdorff s-dimensional measure* is defined by

$$C^s(A) = \sup \{C_0^s(F) : F \subset A,\ F \text{ closed}\}.$$

The second step in the definition of $C^s(A)$ is due to the lack of monotonicity of C_0^s (see [29] and [30, Example 4]). However, in [30], it was shown that the second step can be omitted when restricting oneself to self-similar sets with OSC.

With regard to metric measures based on packings, the standard packing measure P^s (see [28] and [31]) is also defined in a two-step process,

$$P_0^s(A) = \lim_{\delta\to 0} \sup \left\{ \sum_{i=1}^{\infty} (2d_i)^s : 2d_i \leq \delta,\ i = 1, 2, \dots \right\},$$

where the supremum is taken over all *packings* $\{B(x_i, d_i)\}_{i\in\mathbb{N}^+}$, with $x_i \in A$ for all i, and $B(x_i, d_i) \cap B(x_j, d_j) = \varnothing$ for $i \neq j$. Then,

$$P^s(A) = \inf \left\{ \sum_{i=1}^{\infty} P_0^s(F_i) \right\},$$

where the infimum is taken over all coverings $\{F_i\}_{i\in\mathbb{N}^+}$ of A by closed sets F_i (cf. [32]). In [33], it was proved that if A is a compact set with $P_0^s(A) < \infty$, then $P^s(A) = P_0^s(A)$, so this simplification applies to any compact subset of a self-similar set with OSC.

The *spherical s-dimensional Hausdorff measure,* $\mathcal{H}^s_{Sph}(A)$, is obtained by removing in (20) the requirement that the balls are centred at points of A. The classical *s-dimensional Hausdorff measure,*

$\mathcal{H}^s(A)$, results if coverings of A by arbitrary subsets, $\{U_i\}_{i\in\ \mathbb{N}^+}$, are considered and $2d_i$ is replaced in (20) with the diameter of U_i, $|U_i|$ (see [34] and [7]). No second step is required for these last two measures.

The packing and the centred Hausdorff measures have a much simpler expression when dealing with self-similar sets E that satisfy the OSC as the browse for optimal packings or coverings can be reduced to the search for optimal density balls within the class of typical balls, $\mathcal{B}$ (see Definition 3). In particular, for any self-similar E that satisfies the OSC and with similarity dimension s, it is known (see [35]) that

$$P^s(E) = \left(\inf\left\{\theta^s_\mu(x,d) : B(x,d)\in\mathcal{B}\right\}\right)^{-1}, \tag{21}$$

and, Lemma (13) of Sec. 3.2 implies that

$$C^s(E) = \left(\sup\left\{\theta^s_\mu(x,d) : B(x,d)\in\mathcal{B}\right\}\right)^{-1}. \tag{22}$$

3 Local structure and typical balls

Now we shall study the local structure of a self-similar set E that satisfies the OSC for a feasible open set $\mathcal{O}$ through the study of the scenery flow of $\alpha\in\mathcal{M}_{\mathcal{S}}(E)$ at a.e. $x\in E$, and through the characterisation of the spectrum of the spherical s-densities of measures in $\mathcal{M}^s\lfloor_E$ (Sec. 3.2), a limiting set that helps to summarise the structure in the neighbourhood of a point (Sec. 1.3).

3.1 Scenery flow and tangent measures

We start by giving details on the construction of $\widetilde{Tan}(\nu,x)$ for $\nu\in\mathcal{M}_{\mathcal{S}}(E)$ and $x\in E$ (see (18) for notation).

3.1.1 Tangent measures, identifications and topologies.

Recall that the construction of the sets $\widetilde{Tan}(\nu,x)$ and $\widetilde{Tan}^{st}(\nu,x)$ employs the identification, in the set $\mathcal{M}(\mathbb{R}^n)$, of those measures that are equal up to isometries or mutual multiples (see (4), (5), (6) and (7) for notation). We now examine the construction of the spaces of equivalence classes of tangent measures above in more detail.

For $\nu \in \mathcal{M}(\mathbb{R}^n)$ and $x \in spt(\nu)$, we first consider sequences $\{c_n \nu_{x,t_n}\}_{n=0}^{\infty}$, where for every $n \in \mathbb{N}$, $c_n > 0$,

$$\nu_{x,t_n} := \frac{1}{\nu(B(x, dt_n^{-1}))} (g_{x,t_n})_\sharp \, \nu\lfloor_{B(x,dt_n^{-1})}, \tag{23}$$

$d \leq 1$ and g_{x,t_n} is some similarity with expanding ratio t_n that maps the ball $B(x, t_n^{-1})$ onto the ball $B(z_n, 1)$, with $z_n = g_{x,t_n}(x)$, so each ν_{x,t_n} is a probability measure supported on $B(z_n, d)$. Then, $\widetilde{Tan}(\nu, x)$ and $\widetilde{Tan}^{st}(\nu, x)$ consist of the equivalence classes of non-null weak and strong limits, respectively, as $t_n \to \infty$, of such sequences $\{c_n \nu_{x,t_n}\}_{n=0}^{\infty}$ (see (7)). Lemma 8 shows that the elements in $\widetilde{Tan}(\nu, x)$ and $\widetilde{Tan}^{st}(\nu, x)$ do not depend on either the sequence of constants c_n or the particular elements chosen in the equivalence classes $\widetilde{\nu_{x,t_n}}$ as long as the convergence of these elements is guaranteed.

Remark 7 *The unit ball D does not play any essential role in our definition of tangent measures in the quotient space $\widetilde{\mathcal{M}}(\mathbb{R}^n)$. In the opposite direction (second approach) we may, in a more akin way to the classical approach, require the similarities g_{x,t_n} to map $B(x, t_n^{-1})$ onto $B(0, 1)$, and then define $Tan_D(\nu, x)$ and $Tan_D^{st}(\nu, x)$ as weak and strong limits in $\mathcal{M}(D)$, respectively, of sequences of such measures ν_{x,t_n}, and $\widetilde{Tan}(\nu, x)$, $\widetilde{Tan}^{st}(\nu, x)$ as the sets of equivalence classes of measures in $Tan_D(\nu, x)$ and $Tan_D^{st}(\nu, x)$, respectively.*

This second method gives spaces of tangent equivalence classes which are particular cases of these in our primary approach. Are these equivalent methods? In order to answer this question, let a sequence $\{c_n \nu_{x,t_n}\}_{n=0}^{\infty}$, as in (23), converge to a non-null Radon measure α. By Lemma 8 we may assume $c_n = 1$ for all $n \in \mathbb{N}^+$. Since the measures ν_{x,t_n} are supported on balls $B(z_n, d)$ with $d \leq 1$ (see Theorem 12 (i)), the limiting measure α must also be supported on a ball $B(z, d)$ with $z_n \underset{n\to\infty}{\longrightarrow} z$. Each measure $\nu'_{x,t_n} = (\tau_{z_n})_\sharp \, \nu_{x,t_n}$, where $\tau_{z_n}(y) = y - z_n$, is equivalent by translation to ν_{x,t_n}, and ν'_{x,t_n} is supported on D. It is easy to see that $\nu_{x,t_n} \xrightarrow[n\to\infty]{w} \alpha$ implies that $\nu'_{x,t_n} \xrightarrow[n\to\infty]{w} \alpha' = (\tau_z)_\sharp \, \alpha$, so α' is equivalent to α and supported on D. Thus, the second method gives the same space $\widetilde{Tan}(\nu, x)$ than our primary method. But $\nu_{x,t_n} \xrightarrow[n\to\infty]{st} \alpha$ does not imply that $\nu'_{x,t_n} \xrightarrow[n\to\infty]{st} \alpha'$, so the second method does not produce the same space $\widetilde{Tan}^{st}(\nu, x)$ than our method.

Observe that, if we let $\nu'_{x,t_n} = (\tau_z)_\sharp \, \nu_{x,t_n}$, then $\nu_{x,t_n} \xrightarrow[n\to\infty]{st} \alpha$ does imply $\nu'_{x,t_n} \xrightarrow[n\to\infty]{st} \alpha' = (\tau_z)_\sharp \, \alpha$. But now the measure ν'_{x,t_n} is supported on the ball $B(z_n - z, d)$ rather than on D. This observation is useful *because D and all the balls $B(z_n - z, d)$ are contained in some ball $B(0,R)$ for R large enough (notice that z_n is a convergent sequence of points), so the convergence $\nu_{x,t_n} \underset{n\to\infty}{\longrightarrow} \alpha$ (weak or strong) occurs in $\mathcal{M}(B(0,R))$, and we can see that, if we consider vague convergence of measures, we do not obtain anything new, since in the Polish space $B(0,R)$ both convergences are equivalent ([15], Appendix).*

Lemma 8

(i) The sequences $\{c_n\}_{n=0}^{\infty}$ in the construction of $\widetilde{Tan}(\nu,x)$ and $\widetilde{Tan}^{st}(\nu,x)$ can be taken to be $c_n = 1$, $n = 0,1,2,...$

(ii) Let $\nu \in \mathcal{M}(\mathbb{R}^n)$, $x \in spt(\nu)$ and $\alpha \in Tan(\nu,x)$. Let $\{t_n\}_{n=0}^{\infty} \uparrow \infty$ be such that $\{\nu_{x,t_n}\}_{n=0}^{\infty} \xrightarrow[n\to\infty]{w} \alpha$. Assume also that there is a sequence $\{f_n\}_{n=0}^{\infty}$ in the set of isometries $\mathcal{I}_n$ such that $\{(f_n)_\sharp \nu_{x,t_n}\}_{n=0}^{\infty} \xrightarrow[n\to\infty]{w} \alpha'$. Then, there is $f \in \mathcal{I}_n$ such that $(f)_\sharp\alpha = \alpha'$. The same is true if the convergence holds in the topology of the strong convergence in $\mathcal{M}(\mathbb{R}^n)$.

Proof.

(i) By definition, a weak limiting measure α as in (7) is a non-null measure in $\mathcal{M}(\mathbb{R}^n)$. Therefore, the sequence of constants $\{c_n\}$ must be bounded above and below by two positive and finite constants. We can choose a subsequence $\{c_{n_k}\}_{k=0}^{\infty}$ that converges to a constant c, and then the whole sequence $c\nu_{x,t_n}$ must converge to the weak limit α. This gives $\nu_{x,t_n} \xrightarrow[n\to\infty]{w} c^{-1}\alpha$, which belongs to the same equivalence class in $\widetilde{Tan}(\nu,x)$ as α. On the other hand, the non-null weak limits in $\mathcal{M}(\mathbb{R}^n)$ of sequences $\{\nu_{x,t_n}\}_{n=0}^{\infty}$ are particular cases of those of sequences $\{c_n\nu_{x,t_n}\}$. This completes the proof of part (i) for weak limits. The argument also holds true for strong limits.

(ii) For any $n \in \mathbb{N}^+$, we can write $f_n(\cdot) = g_n(\cdot) + a_n$, where g_n is an orthogonal map and $a_n \in \mathbb{R}^n$. Recall that ν_{x,t_n} is supported on $B(z_n,d)$, so $(g_n + a_n)_\sharp(\nu_{x,t_n})$ is supported on $a_n + B(g_n(z_n),d)$ ([7], Theorem 1.18), with $z_n \underset{n\to\infty}{\longrightarrow} z$. As $\|g_n(z_n)\| = \|z_n\|$, this means that, if $\nu'_{x,t_n} := (f_n)_\sharp \nu_{x,t_n}$ converges, in the weak topology of $\mathcal{M}(\mathbb{R}^n)$, to some non-null measure α' in $\mathcal{M}(\mathbb{R}^n)$, the sequence a_n must be bounded, and then the sequence $\{f_n\}_{n=0}^{\infty}$ is also bounded in the supremum norm. Therefore, there is a convergent subsequence, $\{f_{n_k}\}_{k=0}^{\infty}$, of $\{f_n\}_{n=0}^{\infty}$. Let $f := \lim_{k\to\infty} f_{n_k}$. Since the sequence $\{(f_{n_k})_\sharp(\nu_{x,t_{n_k}})\}_{k=0}^{\infty}$ converges to α', we have that

$$\alpha' = \lim_{k\to\infty} f_{n_k\sharp}(\nu_{x,t_{n_k}}) = f_\sharp\alpha, \tag{24}$$

which proves that $\alpha' \cong \alpha$. The second equality in (24) holds true because, for any φ in the space $C_0(\mathbb{R}^n)$ of continuous, compactly supported functions on $\mathbb{R}^n$ and for any $\varepsilon > 0$, there is $k_0 > 0$ such that for $k \geq k_0$, we have

$$\|\varphi \circ f_{n_k} - \varphi \circ f\| \leq \frac{\varepsilon}{2},$$
$$\left\| \int \varphi \circ f \; d(\nu_{x,t_{n_k}}) - \int \varphi \circ f \; d\alpha \right\| \leq \frac{\varepsilon}{2},$$

and then

$$\left\| \int \varphi \; d\left(f_{n_k \sharp}(\nu_{x,t_{n_k}}) \right) - \int \varphi \; d(f_\sharp \alpha) \right\|$$
$$\leq \int \|\varphi \circ f_{n_k} - \varphi \circ f \; \| \, d(\nu_{x,t_{n_k}}) + \left\| \int \varphi \circ f \; d(\nu_{x,t_{n_k}}) - \int \varphi \circ f \; d\alpha \right\| \leq \varepsilon.$$

If $\{\nu'_{x,t_n}\}_{n=0}^{\infty}$ converges to α' in the strong topology of $\mathcal{M}(D)$, then it also converges in the weak topology and the argument above applies. ■

3.1.2 Scaling properties of typical balls and scenery flow

We need some preliminary lemma and the following definition.

Definition 9 *Given a measure $\alpha \in \mathcal{M}^s\lfloor_E$, two Euclidean balls $B(x,d)$ and $B(x',d')$ are said to be α-density equivalent if $\theta^s_\alpha(x,d) = \theta^s_\alpha(x',d')$.*

We start with two elementary scaling properties of typical balls for measures in $\mathcal{M}_\mathcal{S}(E)$ and in $\mathcal{M}^s\lfloor_E$.

Lemma 10 *Let E be a self-similar set generated by the system $\Psi = \{f_i\}_{i \in M}$ of similarities of $\mathbb{R}^n$, with $M = \{0, 1, \ldots, m-1\}$, and similarity dimension s. Let $\mathcal{O}$ be a feasible open set (for Ψ) and let $i \in M^*$. Then*

(i)

$$\mu_\mathbf{p}(f_i(A)) = p_i \mu_\mathbf{p}(A), \text{ for } \mu_\mathbf{p} \in \mathcal{M}_\mathcal{S}(E) \quad \text{and } \mu_\mathbf{p}\text{-measurable } A \subset \mathcal{O}, \tag{25}$$

(ii)

$$\mu_\mathbf{p}(f_i^{-1}(C)) = p_i^{-1} \mu_\mathbf{p}(C) \text{ for } \mu_\mathbf{p} \in \mathcal{M}_\mathcal{S}(E) \text{ and } \mu_\mathbf{p}\text{-measurable } C \subset \mathcal{O}_i, \tag{26}$$

(iii)

$$B(f_i(x), r_i d) \text{ is } \alpha\text{-density equivalent to } B(x,d) \text{ for } \alpha \in \mathcal{M}^s\lfloor_E \text{ and } B(x,d) \subset \mathcal{O}, \tag{27}$$

(iv)

$$f_i^{-1}(B(x,d)) \text{ is } \alpha\text{-density equivalent to } B(x,d) \text{ for } \alpha \in \mathcal{M}^s\lfloor_E \text{ and } B(x,d) \subset \mathcal{O}_i. \tag{28}$$

Proof. The proof of (25) is trivial from (16) if E satisfies SSC. If SSC does not hold, then

$$\mu_{\mathbf{p}}(f_j^{-1}(f_i(A))) \leq \mu_{\mathbf{p}}(\partial\mathcal{O}) = 0 \text{ for } j \neq i,$$

because $A \subset \mathcal{O}$ and, hence, $f_j^{-1}(f_i(A)) \cap E \subset \partial\mathcal{O}$, which is known to be a $\mu_{\mathbf{p}}$- null set (cf. [27]), so (25) also follows from (16). If we set $A = f_i^{-1}(C)$ in (25), we obtain (26) (see also [11]). By (19), we can apply (25) and (26) to any measure $\alpha \in \mathcal{M}^s\lfloor_E$, which easily gives (27) and (28). ■

Before stating the main theorem of this section, we will see the following lemma.

Lemma 11

(i) Let $g,\ f : \mathbb{R}^n \to \mathbb{R}^n$, $\alpha \in \mathcal{M}(\mathbb{R}^n)$, $\lambda > 0$ *and* $A \subset \mathbb{R}^n$ *be an* α*-measurable subset. Then, the following equalities hold true:*

- $\lambda\,(g)_\sharp\,(\alpha) = (g)_\sharp\,(\lambda\alpha)$,
- $(f \circ g)_\sharp\alpha = f_\sharp(g)_\sharp(\alpha)$, *and*
- $(g_\sharp\alpha)\lfloor_A = g_\sharp(\alpha\lfloor_{g^{-1}(A)})$

(ii) Let α *be a measure on* $\mathcal{M}(\mathbb{R}^n)$, $g : \mathbb{R}^n \to \mathbb{R}^n$ *a bijective mapping and* $\beta := g_\sharp\,(\alpha)$. *Then,* $\alpha = (g^{-1})_\sharp\beta$.

(iii) If $\{\alpha_k\}_{k\in\mathbb{N}}$ *is a sequence of measures on* $\mathcal{M}(\mathbb{R}^n)$ *and* $(g)_\sharp\,(\alpha_k) \xrightarrow[k\to\infty]{st} \beta$, *then* $\alpha_k \xrightarrow[k\to\infty]{st} \left(g^{-1}\right)_\sharp(\beta)$.

(iv) Let $B(x_n, d) := B_n$ *be a sequence of closed balls that converges in the Hausdorff metric to a closed ball* $B(x,d) := B$, *and let* $\alpha \in \mathcal{M}(B)$ *with* $\alpha(\partial B) = 0$. *Then* $\alpha\lfloor_{B_n} := \alpha_n \xrightarrow[n\to\infty]{st} \alpha$.

Proof.

Parts (i)-(iii) easily follows from the definitions.

Recall that $\alpha_n \xrightarrow[n\to\infty]{st} \alpha$ means that $\alpha_n(A) \underset{n\to\infty}{\longrightarrow} \alpha(A)$ for any Borel set $A \subset \mathbb{R}^n$. Let $\alpha \in \mathcal{M}(B)$ and let K be any compact set contained in the interior U of B. The distance $d(K, \partial B) = \min\{\|x-y\| : x \in K,\ y \in \partial B\}$ must be a quantity $\varepsilon > 0$ and then $K \subset B(x, d-\varepsilon)$. The convergence of B_n to B implies that there is an $n_0 \in \mathbb{N}^+$ such that, for $n > n_0$, $\|x - x_n\| \leq \varepsilon$. Then, if $z \in K$,

$$\|z - x_n\| \leq \|z - x\| + \|x - x_n\| \leq d,$$

which shows that $K \subset B \cap B_n$ for $n > n_0$. Then, for such values of n, we have

$$\alpha_n(K) = \alpha(B_n \cap K) = \alpha(K)$$

We now prove that $\alpha_n \xrightarrow[n\to\infty]{st} \alpha$ also holds in the σ-field $\mathfrak{B}(B)$ of Borel subsets of B. Let

$$\mathcal{A} := \left\{ A \subset B : A \text{ is } \alpha\text{-measurable and } \lim_{n\to\infty} \alpha_n(A) = \alpha(A) \right\}.$$

(Notice that any α-measurable set is also α_n-measurable for all $n \in \mathbb{N}^+$). It is easy to check that $B \in \mathcal{A}$, that $B - A := A^c \in \mathcal{A}$ if $A \in \mathcal{A}$, and that $\mathcal{A}$ is closed under a finite union of its members or, in short, that $\mathcal{A}$ is a field. Let F_k be a sequence of members of $\mathcal{A}$. In order to show that $\cup_{k\in\mathbb{N}^+} F_k \in \mathcal{A}$, we first write $\cup_{k\in\mathbb{N}^+} F_k = \cup_{k\in\mathbb{N}^+} G_k$, where $G_k = \cup_{i=1}^k F_i$. This shows that $\cup_{k\in\mathbb{N}^+} F_k$ can be expressed as a countable union of the increasing sequence G_k of members of $\mathcal{A}$. Furthermore, $\cup_{k\in\mathbb{N}^+} F_k = \cup_{k\in\mathbb{N}^+} H_k$, where $H_k = (G_k - G_{k-1})$ with $G_0 = \varnothing$. Now, each $H_k \in \mathcal{A}$ and $H_k \cap H_{k'} = \varnothing$ for $k \neq k'$. Then, using that each α_n is a measure, we have

$$\lim_{n\to\infty} \alpha_n \left(\bigcup_{k\in\mathbb{N}^+} H_k \right) = \sum_{k\in\mathbb{N}^+} \lim_{n\to\infty} \alpha_n(H_k) = \sum_{k\in\mathbb{N}^+} \alpha(H_k) = \alpha \left(\bigcup_{k\in\mathbb{N}^+} H_k \right).$$

This completes the proof that $\mathcal{A}$ is a σ-field. Notice that any closed set $K \subset B$ can be written as the union of the α and α_n-null set $K \cap \partial B$ and of the set $K - \partial B$, which belongs to $\mathcal{A}$ as a countable union of compact sets $K \cap B(x, d - n^{-1}) \subset U$. Thus, the class $\mathcal{K}$ of closed subsets of B is contained in $\mathcal{A}$. We know that the σ-fields generated by $\mathcal{K}$ and by $\mathcal{A}$ satisfy $\mathfrak{B}(B) = \sigma(\mathcal{K}) \subset \sigma(\mathcal{A}) = \mathcal{A}$. This gives the strong convergence of α_n to α on $\mathfrak{B}(B)$. ■

We can now go to the scenery flow of measures in $\mathcal{M}_{\mathcal{S}}(E)$.

Theorem 12 *Let E be a self-similar set generated by the system $\Psi = \{f_i\}_{i\in M}$ of similarities on $\mathbb{R}^n$, with $M = \{0, 1, \ldots, m-1\}$ and similarity dimension s. Let $\mathcal{O}$ be a feasible open set (for Ψ) and $\mu_{\mathbf{p}} \in \mathcal{M}_{\mathcal{S}}(E)$. Then, for any $\mu_{\mathbf{p}}$-measurable set $B \subset \mathcal{O}$ and $i \in M^*$, the following statements hold true.*

(i)

$$\mu_{\mathbf{p}}\lfloor_{B_i} = p_i (f_i)_\sharp (\mu_{\mathbf{p}}\lfloor_B).$$

(ii)

$$\mu_{\mathbf{p}}\lfloor_B = p_i^{-1} \left(f_i^{-1}\right)_\sharp (\mu_{\mathbf{p}}\lfloor_{B_i}). \tag{29}$$

(iii) There is a subset $\widehat{E} \subset E$ with $\mu_{\mathbf{p}}(\widehat{E}) = 1$ such that, if $x \in E$ and $B(x,d) \subset \mathcal{O}$, then for any $y \in \widehat{E}$, there is a sequence $\{i_j\}_{j\in\mathbb{N}^+}$ with $i_j \in M^*$ and a sequence of balls $\{B(y, dr_{i_j})\}_{j\in\mathbb{N}^+}$ such that

$$p_{i_j}^{-1}\left(f_{i_j}^{-1}\right)_\sharp \left(\mu_{\mathbf{p}}\lfloor_{B(y,dr_{i_j})}\right) \xrightarrow[j\to\infty]{st} \mu_{\mathbf{p}}\lfloor_{B(x,d)}$$

(iv) For any $x \in \widehat{E}$,

$$\widetilde{\mathcal{M}}_{\mathcal{S}}(\mathcal{B}) \subset \widetilde{Tan}^{st}(\mu_{\mathbf{p}}, x),$$

where $\mathcal{M}_{\mathcal{S}}(\mathcal{B})$ is defined in (8).

Proof. In order to show (i), let $\mu_{\mathbf{p}} \in \mathcal{M}_{\mathcal{S}}(E)$, $i \in M^*$ and let $B \subset \mathcal{O}$ and $A \subset \mathbb{R}^n$ be $\mu_{\mathbf{p}}$-measurable sets. Then,

$$\begin{aligned}(p_i(f_i)_\sharp(\mu_{\mathbf{p}}\lfloor_B))(A) &= p_i(\mu_{\mathbf{p}}\lfloor_B)\left(f_i^{-1}(A)\right)\\ &= p_i\mu_{\mathbf{p}}(f_i^{-1}(A\cap B_i)) = \mu_{\mathbf{p}}\lfloor_{B_i}(A),\end{aligned}$$

where the third equality follows from (26) and (i) is proved. Analogously, (ii) follows from (25).

Now, let

$$\widehat{E} = \left\{y \in E : \left\{\mathcal{T}^k(y) : k \in \mathbb{N}^+\right\} \text{ is dense in } E\right\} \tag{30}$$

(see (15) in Sec. 2 the definition of $\mathcal{T}$). It is well known (cf. [36]) that the set $\widehat{E}$ has a full $\mu_{\mathbf{p}}$-measure. Let $x \in E$, $B(x,d) \subset \mathcal{O}$, $y \in \widehat{E}$ and $\{x_j\}_{j\in\mathbb{N}^+}$ such that $\lim_{j\to\infty} x_j = x$ (in the Euclidean metric) with $x_j \in \mathcal{T}^{k_j}(y)$ for every $j \in \mathbb{N}^+$. We may also assume that $B(x_j,d) \subset \mathcal{O}$ for every $j \in \mathbb{N}^+$. We shorten $B(x_j,d)$ to B^j and $B(x,d)$ to B. Since $\lim_{j\to\infty} x_j = x$, it follows that $\left\{B^j\right\}_{j\in\mathbb{N}}$ converges to B in the Hausdorff metric. Also, $\mu_{\mathbf{p}}(\partial B) = 0$ because $\mu_{\mathbf{p}} \in \mathcal{M}_{\mathcal{S}}(E)$ (cf. [37]). Then, Lemma 11 (iv) implies that

$$\mu_{\mathbf{p}}\lfloor_{B^j} \xrightarrow[j\to\infty]{st} \mu_{\mathbf{p}}\lfloor_B. \tag{31}$$

Now, notice that, since $x_j \in \mathcal{T}^{k_j}(y)$ for each $j \in \mathbb{N}$, there is $i_j \in M^{k_j}$ such that $f_{i_j}(x_j) = y$ (see Remark 6). Then, $f_{i_j}^{-1}(B(y, dr_{i_j})) = B^j$. By (29) applied to B^j and $i_j \in M^*$, we see that

$$\mu_{\mathbf{p}}\lfloor_{B^j} = p_{i_j}^{-1}\left(f_{i_j}^{-1}\right)_\sharp\left(\mu_{\mathbf{p}}\lfloor_{B(y,dr_{i_j})}\right), \tag{32}$$

which concludes the proof of (iii).

Observe that, in the terminology of Sec. 3.1.1, the right hand term in (32) is, $c_{t_j}\nu_{y,t_j}$ for $\nu = \mu_{\mathbf{p}}$,

$t_j = r_{i_j}^{-1}$ and $c_{t_j} = p_{i_j}^{-1}\mu_{\mathbf{p}}(B(y, dr_{i_j}))$ (recall that ν_{y,t_j} was a normalised blowup and notice also that we may assume, rescaling E if necessary, that all typical balls have a radius $d \le 1$). So, (31) and (7) give $\widetilde{\mu_{\mathbf{p}}\lfloor_B} \in \widetilde{Tan}^{st}(\mu_{\mathbf{p}}, x)$ and part (iv) is proved. ■

3.2 Asymptotic spectra and measure-exact self-similar sets

We shall write $Im(\theta^s_\alpha, \mathcal{B})$ to designate the set

$$Im(\theta^s_\alpha, \mathcal{B}) := \{\theta^s_\alpha(x,d) : B(x,d) \in \mathcal{B}\}$$

(see notation in Definition 3), which plays a relevant role in the geometric analysis of E (see (21) and the lemma below).

Lemma 13 *Let E be a self-similar set generated by the system of similarities of $\mathbb{R}^n$, $\Psi = \{f_i\}_{i\in M}$, with $M = \{0, 1, \ldots, m-1\}$, and similarity dimension s. If E satisfies the OSC, then*

(i)

$$C^s(E) = \left(\sup\left\{\theta^s_\mu(x,d) : B(x,d) \in \mathcal{B}_{\mathcal{O}}\right\}\right)^{-1},$$

where $\mathcal{B}_{\mathcal{O}} := \{B(x,d) \in \mathcal{B} : \ B(x,d) \subset \mathcal{O}\}$ and $\mathcal{O}$ is any feasible open set for Ψ.

(ii)

$$C^s(E) = \left(\sup Im(\theta^s_\mu, \mathcal{B})\right)^{-1}.$$

Proof. It is known that for a general self-similar set that satisfies the OSC (see [35] and [30]),

$$C^s(E) = \left(\sup\{\theta^s_\mu(x,d) : x \in E \text{ and } d > 0\}\right)^{-1} \tag{33}$$

holds. Let $\mathcal{O}$ be any feasible open set. Then, it is enough to show that

$$\sup_{(x,d)\in E\times\mathbb{R}^+} \theta^s_\mu(x,d) \le \sup_{B(x,d)\in\mathcal{B}_{\mathcal{O}}} \theta^s_\mu(x,d).$$

Should this not be the case, there would exist $(x_0, d_0) \in E \times \mathbb{R}^+$ such that $B(x_0, d_0) \notin \mathcal{B}_{\mathcal{O}}$ and

$$\theta^s_\mu(x_0, d_0) > \sup_{B(x,d)\in\mathcal{B}_{\mathcal{O}}} \theta^s_\mu(x,d).$$

In order to show that this contradicts (33), take $x^* \in E \cap \mathcal{O}$ such that there is $i \in M^*$ with $f_i(x^*) = x^*$. Let $\rho_1 := \min\{\|x^* - z\| : z \in \partial\mathcal{O}\}$. Observe that, if we take $\rho_2 > 0$ so that $B(x_0, d_0) \subset B(x^*, \rho_2)$ and

$k \in \mathbb{N}^+$, satisfying that $r_i^k \rho_2 < \rho_1$, then

$$f_i^k(B(x_0, d_0)) \subset f_i^k(B(x^*, \rho_2)) = B(x^*, r_i^k \rho_2) \subset \mathcal{O},$$

which, using that $f_i^k(B(x_0, d_0) \cap S) \subset f_i^k(B(x_0, d_0)) \cap S$, raises the contradiction

$$\theta_\mu^s(x_0, d_0) \leq \frac{r_i^{-ks} \mu(f_i^k(B(x_0, d_0)))}{(2d_0)^s} = \frac{\mu(B(f_i^k(x_0), r_i^k d_0))}{(2d_0 r_i^k)^s} \leq \sup_{B(x,d) \in \mathcal{B}_\mathcal{O}} \theta_\mu^s(x, d).$$

Part (ii) is trivial from (i). ■

In the next theorem, we shall establish the relationships between the pointwise and global spectra, the set $Im(\theta_\alpha^s, \mathcal{B})$ and its extreme values $\alpha(E)\,(P^s(E))^{-1}$ and $\alpha(E)\,(C^s(E))^{-1}$.

Theorem 14 *Let $E \subset \mathbb{R}^n$ be a self-similar set that satisfies the SOSC with feasible open set $\mathcal{O}$ and similarity dimension s, and let $\alpha \in \mathcal{M}^s\lfloor_E$. Then, the following statements hold true.*

(i) For $x \in E$, it holds that

$$Spec(\alpha, x) = \left[\underline{\theta}_\alpha^s(x), \overline{\theta}_\alpha^s(x)\right]$$

(see (13) and (12) for notation)

$$Spec(\alpha, E) \subset [\kappa_1, \kappa_2]$$

with $0 < \kappa_1 \leq \kappa_2 < \infty$.

(ii) There is a subset $\widehat{E} \subset E$ with $\mu(\widehat{E}) = 1$ such that, for any $y \in \widehat{E}$,

$$Spec(\alpha, y) = Spec(\alpha, \widehat{E}) = Spec(\alpha, \mathcal{O} \cap E).$$

(iii)

$$\left(\frac{\alpha(E)}{P^s(E)}, \frac{\alpha(E)}{C^s(E)}\right) \subset Im(\theta_\alpha^s, \mathcal{B}) \subset Spec(\alpha, \mathcal{O} \cap E) \subset \left[\frac{\alpha(E)}{P^s(E)}, \frac{\alpha(E)}{C^s(E)}\right].$$

Proof. That $\underline{\theta}_\alpha^s(x)$ and $\overline{\theta}_\alpha^s(x)$ belong to and are the extreme values of $Spec(\alpha, x)$ follows from the definitions. That all the intermediate values in between also belong to $Spec(\alpha, x)$ is a consequence of the continuousness of $\theta_\alpha^s(x, d)$, with respect to d. This last property follows from the fact that the α-measure of the boundary of Euclidean balls is always null [37] for any measure $\alpha \in \mathcal{M}_\mathcal{S}(E)$, which proves the first assertion of (i). The second assertion is well known [2].

In order to prove (ii), let $\widehat{E}$ be the full μ-measure subset of points of E that have a dense geometric shift orbit in E (see (30)) and let $y \in \widehat{E}$. The inclusions $Spec(\alpha, y) \subset Spec(\alpha, \widehat{E}) \subset Spec(\alpha, \mathcal{O} \cap E)$ are trivial as $\widehat{E} \subset \mathcal{O}$. This follows from the fact that, if $y \notin \mathcal{O}$, then $\mathcal{T}(y) \cap \mathcal{O} = \varnothing$ because $f_i(\mathcal{O}) \subset \mathcal{O}$ for any $i \in M$, and repeating the same argument, we see that $T^k(y)$ could not be dense in E.

The corresponding equalities would follow if we prove $Spec(\alpha, \mathcal{O} \cap E) \subset Spec(\alpha, y)$. This holds true because, if $z = \lim_{k\to\infty} \theta^s_\alpha(x, d_k)$ for $x \in \mathcal{O}{\cap}E$ and $d_k \underset{k\to\infty}{\longrightarrow} 0$, since $B(x, d_k) \in \mathcal{B}$ for any sufficiently large k, we can apply Theorem 12 (iii) to see that, for such values of k, $\theta^s_\alpha(x, d_k) \in Spec(\alpha, y)$ and, hence, $\lim_{k\to\infty} \theta^s_\alpha(x, d_k) \in Spec(\alpha, y)$ easily follows from (i). This ends the proof of (ii).

Finally, the first inclusion in (iii) for $\alpha = \mu$ follows from the continuousness of the function $\theta^s_\mu(x, d)$ on $\mathbb{R}^n \times \mathbb{R}^+$ since

$$\frac{1}{P^s(E)} \leq \theta^s_\mu(x, d) \leq \frac{1}{C^s(E)}$$

holds if $B(x, d) \in \mathcal{B}$ as a straightforward consequence of (21) and (22). The arguments given in the proof of (ii) applied to μ show that, if $B(x, d) \in \mathcal{B}$, then $\theta^s_\mu(x, d) \in Spec(\mu, \mathcal{O} \cap E)$, which gives the next inclusion in (iii). The last inclusion follows from the observation that $Spec(\mu, \mathcal{O}{\cap}E)$ consists of limiting values of sequences with terms in $Im(\theta^s_\mu, \mathcal{B})$, whose extreme values are $\frac{1}{P^s(E)}$ and $\frac{1}{C^s(E)}$. Using (19), we get that $\theta^s_\alpha(x, d) = \alpha(E)\theta^s_\mu(x, d)$, and (iii) follows for any $\alpha \in \mathcal{M}^s\lfloor_E$. ■

Of note is the case in which the extreme values of $\theta^s_\alpha(x, d)$ are attained on $\mathcal{B}$. In this case, we have the following result.

Corollary 15 *Let $\alpha \in \{\mu,\ P^s\lfloor_E,\ C^s\lfloor_E\}$. Under the hypotheses of Theorem 14, if there are two balls $B(x_1, d_1)$ and $B(x_2, d_2)$, both in $\mathcal{B}$, such that*

$$\theta^s_\mu(x_1, d_1) = \inf\left\{\theta^s_\mu(x, d) : B(x, d) \in \mathcal{B}\right\} \tag{34}$$

and

$$\theta^s_\mu(x_2, d_2) = \sup\left\{\theta^s_\mu(x, d) : B(x, d) \in \mathcal{B}\right\}, \tag{35}$$

the inclusions in Theorem 14 (iii) can be replaced with equalities.

Proof. The first inclusion in Theorem 14 (iii), together with (21), (22), (34) and (35), implies that

$$Im(\theta^s_\mu, \mathcal{B}) = \left[\frac{1}{P^s(E)}, \frac{1}{C^s(E)}\right],$$

which, in turn, gives that $Im(\theta^s_\alpha, \mathcal{B}) = Spec(\alpha, \mathcal{O}{\cap}E)$. ■

Corollary 15 motivates the introduction of the class of α-*exact self-similar sets* with special properties.

Definition 16 *We say that the self-similar set E is α-*exact *if there exists $B \in \mathcal{C}_\alpha$ such that*

$$\frac{\mu(B)}{|B|^s} = \sup\left\{\frac{\mu(B)}{|B|^s} : B \in \mathcal{C}_\alpha\right\}$$

if $\alpha \in \{C^s\lfloor_E, \mathcal{H}^s\lfloor_E, \mathcal{H}_{Sph}\lfloor_E\}$, and

$$\frac{\mu(B)}{|B|^s} = \inf\left\{\frac{\mu(B)}{|B|^s} : B \in \mathcal{C}_\alpha\right\},$$

if $\alpha = P^s\lfloor_E$, where $\mathcal{C}_\alpha$ is what we call "the relevant class of sets" for the measure α, which is defined as

- $\mathcal{C}_\alpha := \mathcal{B}$ *if* $\alpha \in \{P^s\lfloor_E, C^s\lfloor_E\}$,
- $\mathcal{C}_{\mathcal{H}^s\lfloor_E} := \{B \subset \mathbb{R}^n : B$ *is a convex set*$\}$ *and*
- $\mathcal{C}_{\mathcal{H}^s_{Sph}\lfloor_E} := \{B \subset \mathbb{R}^n : B$ *is a closed ball*$\}$.

One nice property that α-exact self-similar sets have is that they possess optimal coverings or packings, that is, almost-coverings (i.e. coverings for α-almost all points in E) or packings whose s-volume gives the exact value of the corresponding α-measure, whilst if α-exactness is not fulfilled, we can only hope to find coverings or packings with s-volume arbitrarily close to the corresponding α-measure.

Example 17 *Self-similar sets E with the strong separation condition are an example of α-exact self-similar sets. See [38] for $\alpha \in \left\{P^s\lfloor_E,\ \mathcal{H}^s\lfloor_E,\ \mathcal{H}^s_{Sph}\lfloor_E\right\}$ and [30] for $\alpha = C^s$.*

Example 18 *The Sierpinski gasket S is an example of a set where the strong separation condition does not hold, and that is a $P^s\lfloor_S$-exact (see [22]) and $C^s\lfloor_S$-exact (see [23]) set.*

In [39], it is shown a class of self-similar sets E with OSC in the line whose members can be non-$\mathcal{H}^s\lfloor_E$-exact (and, consequently, non-$\mathcal{H}^s_{Sph}\lfloor_E$-exact since these two measures coincide in the line), and the authors find conditions under which they are $\mathcal{H}^s\lfloor_E$-exact.

Example 19 *Self-similar sets E with OSC in the line, with similarity dimension s, and that admit an open interval as a feasible open set, are an example of $P^s\lfloor_E$-exact self-similar sets [40].*

3.3 Complexity of the local structure of self-similar sets

We now show how these results allow us to explore the complexity of the local geometric structure of self-similar sets that satisfy the OSC condition.

First, we need to properly define the equivalence classes of restricted balls. Notice that different Euclidean balls, even if they share the centre, can produce the same restricted balls. This motivates the following definitions that are valid for general subsets of $\mathbb{R}^n$.

Definition 20 *Given a subset $A \subset \mathbb{R}^n$, the spherical diameter of A is defined by*

$$|A|_{Sph} = \inf\{2d : A = A \cap B(x,d) \text{ for some } x \in A\}$$

Definition 21 *Given a subset $A \subset \mathbb{R}^n$, we say that the restricted ball $B(x,d) \cap A$ is proper and write $B(x,d) \cap A \in \mathfrak{P}(A)$ if $x \in A$ and $2d = |B(x,d) \cap A|_{Sph}$.*

Definition 22 *Given a measure α on $\mathbb{R}^n$ and an α-measurable s-set $A \subset \mathbb{R}^n$, we define the α-spherical s-density of A by*

$$\theta^s_{Sph(\alpha)}(A) = \frac{\alpha(A)}{\left(|A|_{Sph}\right)^s}.$$

Definition 23 *Given a subset $A \subset \mathbb{R}^n$ and two restricted balls $B(x,d) \cap A$, $B(x',d') \cap A$ both in $\mathfrak{P}(A)$, we say that they are similarity-equivalent and write $B(x,d) \cap A \simeq_{\mathcal{S}_n} B(x',d') \cap A$ if there is an $f \in \mathcal{S}_n$ such that*

$$B(x',d') \cap A = f(B(x,d) \cap A).$$

Lemma 24 *Let $A \subset \mathbb{R}^n$ and $B(x,d) \cap A \in \mathfrak{P}(A)$.*

(i) If $f \in \mathcal{S}_n$ has similarity constant r_f, then $f(B(x,d)) \cap f(A) \in \mathfrak{P}(f(A))$ and $|f(B(x,d)) \cap f(A)|_p = r_f d$.

(ii) Let $\alpha \in \mathcal{M}^s$ and let A be an α-measurable s-set. If $B(x,d) \cap A \simeq_{\mathcal{S}_n} B(x',d') \cap A$, then

$$\theta^s_{Sph(\alpha)}(B(x,d) \cap A) = \theta^s_{Sph(\alpha)}(B(x',d') \cap A)$$

Proof. Let $A \subset \mathbb{R}^n$, $B(x,d) \cap A \in \mathfrak{P}(A)$. In order to show (i), assume that $f(B(x,d)) \cap f(A)$ is not proper. Then, there is a ball $B(y,\rho)$ such that

$$B(y,\rho) \cap f(A) = f(B(x,d)) \cap f(A)$$

with $y \in f(A)$ and $\rho < r_f d$. Then $B(f^{-1}(y), r_f^{-1}\rho) \cap A = B(x,d) \cap A$ with $f^{-1}(y) \in A$ and $r_f^{-1}\rho < d$, in contradiction with $|B(x,d) \cap A|_{Sph} = 2d$. Therefore, $f(B(x,d)) \cap f(A) \in \mathfrak{P}(f(A))$ and $|f(B(x,d)) \cap f(A)|_{Sph} = 2r_f d$.

Part (ii) is now trivial since $\alpha \in \mathcal{M}^s$ and, hence,

$$\alpha(B(x',d') \cap A) = \alpha(f(B(x,d) \cap A)) = r_f^s \alpha(B(x,d) \cap A)$$

and, by (i),

$$\left(|B(x',d') \cap A|_{Sph}\right)^s = \left(|f(B(x,d) \cap f(A))|_{Sph}\right)^s = r_f^s (2d)^s = r_f^s \left(|B(x,d) \cap A|_{Sph}\right)^s.$$

■

Now we can proceed to state our result for the complexity of the local geometry of self-similar sets with OSC.

Corollary 25 *Under the assumptions of Theorem 14, assume either that s is a non-integer real number, or that s is integer and the self-similar set E satisfies the strong separation condition. Then, there is an uncountable number of equivalence classes in the quotient space $Sph_E/\simeq_{\mathcal{S}_n}$.*

Proof. By Lemma 24 (ii), we know that all restricted balls in an equivalence class of $Sph_E/\simeq_{\mathcal{S}_n}$ share the same μ-spherical s-density, which allows us to naturally define a mapping $\theta_\mu^s : Sph_E/\simeq_{\mathcal{S}_n} \to Im(\theta_\mu^s, \mathcal{B})$. This implies that the inverse $\left(\theta_\mu^s\right)^{-1} : Im(\theta_\mu^s, \mathcal{B}) \to Sph_E/\simeq_{\mathcal{S}_n}$ of such mapping is an injective correspondence.

In the case of non-integer dimension, using Marstrand's Theorem, parts (ii) and (iii) of Theorem 14 and that $\mu(\widehat{E}) = 1 > 0$, it follows that $C^s(E) < P^s(E)$ (notice that from the definitions in Sec. 2.2 it is easy to see that $C^s(E) \leq P^s(E)$). This, together with Theorem 14 (iii), means that $Im(\theta_\mu^s, \mathcal{B})$ contains an interval with uncountably many points and the proof is completed.

In the case in which E satisfies the strong separation condition, we use a result by Hutchinson [2], which shows that if an special separation condition holds, condition that easily follows from the strong separation condition (see Example 3(a) p. 743 in [2]), then E is an unrectifiable self-similar set and, by the rectifiability Preiss's Theorem [3], the $\mathcal{H}^s$-density cannot exist in sets of positive $\mathcal{H}^s$-measure and the argument given above for the non-integer case also applies here. ■

4 The spectrum of the Sierpinski gasket

In this section, we shall apply the results obtained in Theorem 14 to fully characterise the asymptotic spectra of the Sierpinski gasket S.

Recall that the Sierpinski gasket or Sierpinski triangle is a special case of a self-similar set generated by a system $\Psi = \{f_0, f_1, f_2\}$ of three contracting similitudes of the plane, with contraction ratios $r_i := 1/2$, $i \in M$, given by

$$f_0(x,y) = \frac{1}{2}(x,y), \quad f_1(x,y) = \frac{1}{2}(x,y) + (\frac{1}{2},0) \quad \text{and} \quad f_2(x,y) = \frac{1}{2}(x,y) + \frac{1}{2}(\frac{1}{2},\frac{\sqrt{3}}{2}). \tag{36}$$

We shall denote by z_i the fixed point of each f_i, $i = 0,1,2$ that is, $z_0 = (0,0)$, $z_1 = (1,0)$ and $z_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, and by T the equilateral triangle with vertexes z_i, $i \in M$.

It is well known that S is a connected set that satisfies the OSC and has similarity dimension $s = \frac{\log 3}{\log 2}$.

Thanks to previous work on the packing and Hausdorff centred measures of the Sierpinski gasket (cf. [22] and [23]), we know that S is both $P^s\lfloor_S$ and $C^s\lfloor_S$-exact, and we have fairly precise approximations of the values of $P^s(S)$ and $C^s(S)$.

4.1 Theoretical results

Theorem 26 *Let S be the Sierpinski gasket, $\widehat{S} = \left\{y \in S : \left\{\mathcal{T}^k(y) : k \in \mathbb{N}\right\} \text{ is dense in } S\right\}$, $\mathcal{B}$ be the collection of typical balls, $\mathcal{R}$ be a feasible open set for S, and $\alpha \in \mathcal{M}^s\lfloor_S$. Then, the following statements hold true.*

(i)

$$Spec(\alpha, y) = Spec(\alpha, \widehat{S}) = Spec(\alpha, \mathcal{R} \cap S) = Im(\theta^s_\alpha, \mathcal{B}) = \left[\frac{\alpha(S)}{P^s(S)}, \frac{\alpha(S)}{C^s(S)}\right], \; y \in \widehat{S}. \tag{37}$$

(ii) $Spec(\alpha, S)$ is given by the union of two closed intervals of positive length:

$$Spec(\alpha, S) = \left[\underline{\theta}^s_\alpha(z_0), \overline{\theta}^s_\alpha(z_0)\right] \cup \left[\frac{\alpha(S)}{P^s(S)}, \frac{\alpha(S)}{C^s(S)}\right], \tag{38}$$

where $z_0 = (0,0)$. Furthermore,

$$\underline{\theta}^s_\alpha(z_0) = \min\left\{\theta^s_\alpha(z_0, d) : \frac{1}{2} \le d \le 1\right\} \tag{39}$$

and

$$\overline{\theta}^s_\alpha(z_0) = \max\left\{\theta^s_\alpha(z_0,d) : \frac{1}{2} \le d \le 1\right\}. \tag{40}$$

Proof. Our previous work guarantees that S is a P^s-exact (see [22]) and C^s-exact (see [23]) set. Then, the four equalities in (i) follow as a consequence of Theorem 14 and Corollary 15.

In order to prove (38), let $\mathcal{R}_i$, $i \in \{0,1,2\}$ be the three open rhombi composed of the topological interior of the union of the triangle T and its reflection across the edge of T opposite the point z_i, $i \in M$ (see $\mathcal{R}_2$ in Fig. 1). Using that

$$S = \{z_0, z_1, z_2\} \cup (S \cap \cup_{i=0}^{2} \mathcal{R}_i), \tag{41}$$

we obtain

$$\begin{aligned} Spec(\alpha, S) &= Spec(\alpha, S \cap \cup_{i=0}^{2}\mathcal{R}_i) \cup \left(\cup_{i=0}^{2} Spec(\alpha, z_i)\right) = \\ &= \left[\frac{\alpha(S)}{P^s(S)}, \frac{\alpha(S)}{C^s(S)}\right] \cup Spec(\alpha, z_0), \end{aligned}$$

where the last equality follows from (37), (41) and the fact that, by symmetry, $Spec(\alpha, z_i)$ must be identical for $i \in \{0,1,2\}$.

Observe now that, if $d \le 1/2$, then $B(z_0,d) \cap S = B(z_0,d) \cap f_0(S)$. Hence, using that α is an s-dimensional metric measure

$$\begin{aligned} \theta^s_\alpha(z_0,d) &= \frac{\alpha(B(z_0,d) \cap f_0(S))}{(2d)^s} = \frac{\alpha(f_0(B(z_0,2d) \cap S))}{(2d)^s} \\ &= \frac{\alpha(B(z_0,2d) \cap S))}{(4d)^s} = \theta^s_\alpha(z_0,2d). \end{aligned}$$

If $2d \le 1/2$, we can repeat the argument k times until $1/2 \le 2^k d \le 1$ and $\theta^s_\alpha(z_0,d) = \theta^s_\alpha(z_0,2^k d)$. This shows that

$$\min\{\theta^s_\alpha(z_0,d) : 0 \le d \le 1\} = \min\left\{\theta^s_\alpha(z_0,d) : \frac{1}{2} \le d \le 1\right\} = \underline{\theta}^s_\alpha(z_0),$$

where the last equality can easily be checked and, analogously, (40) holds. ■

Remark 27 *Notice that part (i) of Theorem 26 shows that there is a set of full α-measure whose points exhibit a strongly regular behaviour, whereas part (ii) underlines the special local behaviour of the vertexes as the most isolated points in S. However, the set of exceptional points does not consist only of the vertexes as there might be other exceptional points, all of them belonging to the set $\cup_{i=0}^{2}(\mathcal{R}_i \cap S) - \widehat{S}$.*

The pointwise α-density spectrum of such points is contained in $\left[\frac{\alpha(S)}{P^s(S)}, \frac{\alpha(S)}{C^s(S)}\right]$. *The detection and characterisation of the behaviour of these points remains an open issue.*

4.2 Numerical results

Following the structure of the algorithms developed in [22, 23, 41, 42] for the numerical estimation of the metric measures of self-similar sets, the construction of the computational algorithm used in this work in order to approximate the values of $\underline{\theta}^s_\mu(z_0)$ and $\overline{\theta}^s_\mu(z_0)$ relies upon the discrete approximations of both the Sierpinski gasket and its invariant measure μ. Recall that any two measures in $\mathcal{M}^s\lfloor_S$ are mutually multiple of each other (see (19)), so we can obtain $Spec(\alpha, S)$ from $Spec(\mu, S)$ if we know $\alpha(S)$.

The Sierpinski gasket, as the attractor of $\Psi = \{f_0, f_1, f_2\}$ (see (36)), is the unique non-empty compact set that admits the self-similar decomposition $S = F(S)$, where F is the Hutchinson operator defined, for $A \subset \mathbb{R}^2$, by

$$F(A) := f_0(A) \cup f_1(A) \cup f_2(A).$$

It is well-known that, for any non-empty compact subset $A \subset \mathbb{R}^2$, S can be built with an arbitrary level of detail by increasing the iterations k in $F^k(A)$, where $F^k = F \circ F ... \circ F$ is the k-th iterate of the contracting operator F. This is because $F^k(A) \overset{k\to\infty}{\to} S$ in the Hausdorff metric (cf. [2]). Furthermore, if $A \subset S$, then $F^k(A) \subset S$ for any $k \in \mathbb{N}^+$. In particular, if we take $A_1 := \{z_0, z_1, z_2\}$ as the initial compact set, we obtain the set

$$A_k := F^{k-1}(A_1) \subset S, \ k \geq 2, \tag{42}$$

which approximates S at the iteration k of our algorithm.

The relation between the Markov operator and the natural probability measure $\mu_{\mathbf{p}_s}$ given in (17), with $s = \frac{\log 3}{\log 2}$ and $p_i = r_i^s = 3^{-k}$, and (19) leads to the following relation:

$$\mathbf{M}^k_{\mathbf{p}_s}(\alpha) = \frac{1}{3^k} \sum_{i \in M^k} \alpha \circ f_i^{-1} \overset{w}{\to} \mu, \qquad \alpha \in \mathcal{P}(\mathbb{R}^2). \tag{43}$$

If we consider $\mu_1 := \frac{1}{3}(\delta_{z_0} + \delta_{z_1} + \delta_{z_2})$ as an initial measure α in (43), where δ_x is a unit mass at x, then

$$\mu_k := \mathbf{M}^{k-1}_{\mathbf{p}_s}(\mu_1) = \frac{1}{3^{k-1}} \sum_{i \in M^{k-1}} \mu_1 \circ f_i^{-1} = \frac{1}{3^k} \sum_{i \in M^{k-1}} \left(\delta_{f_i(z_0)} + \delta_{f_i(z_1)} + \delta_{f_i(z_2)}\right) \tag{44}$$

is a probability measure supported on $A_k \subset S$ and $\mu_k \overset{w}{\to} \mu$.

The discrete measure μ_k is the approximation of the invariant measure μ that our algorithm takes at iteration k.

Lemmas 28 and 30 (Lemma 28 is proved in [23]), provide precise relationships between the measures μ_k and μ.

Lemma 28

(i) Let $\{S_i : i \in I \subset M^k\}$, $k \in \mathbb{N}^+$, be a collection of k-cylinder sets of S. Then,

$$\mu\left(\bigcup_{i\in I} S_i\right) \leq \mu_k\left(\bigcup_{i\in I} S_i\right)$$

(ii) Let $A \subset S$, $k \in \mathbb{N}^+$, and let $I = \{i \in M^k : S_i \cap A \neq \varnothing\}$. Then,

$$\mu_k(A) \leq \mu\left(\bigcup_{i\in I} S_i\right)$$

Remark 29 *The comparisons between the measures μ and μ_k on collections of cylinders and sets given in the lemma above are passed to enlarged and reduced balls in part (i) of the next lemma. Since our algorithms compute only μ_k-densities of balls with centres in A_k (see (42)) and with some point of A_k in their boundaries, in part (ii) of this lemma we approximate the μ-measure of a ball centred at x with the μ_k-measure of a ball with its same centre and with a point of A_k at its boundary.*
In order to obtain more accurate estimates of $\underline{\theta}^s_\mu(z_0)$ and $\overline{\theta}^s_\mu(z_0)$ (as we also do in [22] and [23] for the estimation of $P^s(S)$ and $C^s(S)$), it is necessary to consider open balls when searching balls of minimal μ_k-density (see (46)), whereas in the search of balls with maximal μ_k-density, the approximating balls must be taken to be closed balls (see (47)). In the definition of $\underline{\theta}^s_\mu(\cdot)$ and $\overline{\theta}^s_\mu(\cdot)$, the use of open or closed balls has no relevance because the μ-measure of the boundary of any ball is null. However, in the case of densities of the discrete measures μ_k, the values obtained in one or the other case do actually matter, mainly if k is not large.

From now on, we shall use the notation $\mathring{B}(x,d) := \{y \in \mathbb{R}^2 : |x-y| < d\}$ and $\mathring{\theta}^s_\alpha$ for the s-density of α defined using open balls.

Lemma 30 *Let $k > 0$, $x \in \mathbb{R}^2$, and $2^{-k} < d \leq \max_{i \in M} \|z_i - x\|$. Then,*

(i) $\mu_k(B(x, d - 2^{-k})) \leq \mu(B(x, d)) \leq \mu_k(\mathring{B}(x, d + 2^{-k}))$

(ii) If $B(x, d) \cap A_k \neq \varnothing$, then there are points y_k and z_k in A_k such that

$$\mu_k\left(\mathring{B}(x, d_{y_k})\right) \leq \mu(B(x, d)) \leq \mu_k(B(x, d_{z_k})),$$

where $d_{y_k} := |y_k - x|$, $d_{z_k} := |z_k - x|$, and $\{d_{y_k}, d_{z_k}\} \in [d - 2^{-k}, d + 2^{-k}]$.

Proof.

(i) Let

$$H_k := \{i \in M^k : B(x, d - 2^{-k}) \cap S_i \neq \varnothing\}$$

For any $i \in H_k$, $S_i \subset B(x, d)$ holds, so $\cup_{i \in H_k} S_i \subset B(x, d)$. Using Lemma 28 (ii), we have

$$\mu_k(B(x, d - 2^{-k})) \leq \mu(\cup_{i \in H_k} S_i) \leq \mu(B(x, d)).$$

Let

$$G_k := \{i \in M^k : S_i \subset \mathring{B}(x, d + 2^{-k})\}.$$

Then, $\mathring{B}(x, d) \cap S \subset \cup_{i \in G_k} S_i$ and $\cup_{i \in G_k} S_i \subset \mathring{B}(x, d + 2^{-k})$. Using Lemma 28 (i), we get

$$\mu(B(x, d)) = \mu(\mathring{B}(x, d) \cap S) \leq \mu(\cup_{i \in G_k} S_i) \leq \mu_k(\cup_{i \in G_k} S_i) \leq \mu_k(\mathring{B}(x, d + 2^{-k}))$$

(ii) Let $d^* = \max_{i \in M} \|z_i - x\|$. If $S \subset B(x, d)$, then $d = d^*$ and $\mu(B(x, d^*)) = 1 = \mu_k(B(x, d^*)) > \mu_k((\mathring{B}(x, d^*))$, so property (ii) holds for $d_{y_k} = d_{z_k} = d^*$. Let us now assume that $S \not\subseteq B(x, d)$. We prove first that

$$F_k := \{i \in M^k : \partial B(x, d) \cap S_i \neq \varnothing\} \neq \varnothing. \tag{45}$$

If $F_k = \varnothing$, then

$$\cup_{i \in M^k} S_i \subset \mathring{B}(x, d) \cup (B(x, d))^c.$$

We know that $(\cup_{i \in M^k} S_i) \cap \mathring{B}(x, d) \neq \varnothing$ because $B(x, d) \cap A_k \neq \varnothing$ and $F_k = \varnothing$, and we also know that $(\cup_{i \in M^k} S_i) \cap (B(x, d))^c \neq \varnothing$ because $S \not\subseteq B(x, d)$ and $F_k = \varnothing$. This contradicts that $\cup_{i \in M^k} S_i$ is a connected set, and (45) must hold.

Using (i), we have that

$$\mu(B(x, d)) \leq \mu_k(B(x, d + 2^{-k})) = \mu_k(B(x, d_{z_k})),$$

where z_k satisfies $d_{z_k} = \|z_k - x\|$ with

$$d_{z_k} = \max\{\|y - x\| : \ y \in A_k \cap B(x, d + 2^{-k})\}.$$

The inequality $d_{z_k} \leq d + 2^{-k}$ is obvious, and $d_{z_k} \geq d - 2^{-k}$ follows because $F_k \neq \varnothing$ and each k-cylinder S_i, $i \in M^k$ contains some point in A_k.

Using the first inequality in (i), we have

$$\mu(B(x, d)) \geq \mu_k(B(x, d - 2^{-k})) = \mu_k(\mathring{B}(x, d_{y_k})),$$

where y_k satisfies $d_{y_k} = \|y_k - x\|$ with

$$d_{y_k} = \min\{\|y - x\| : \ y \in A_k \cap \left(B(x, d - 2^{-k})\right)^c\}.$$

The inequality $d_{y_k} \geq d - 2^{-k}$ is obvious, and $d_{y_k} \leq d + 2^{-k}$ follows because $F_k \neq \varnothing$. ■

Theorem 26 allows us to characterise $Spec(\alpha, S)$ for $\alpha \in \{\mu,\ P^s\lfloor_S,\ C^s\lfloor_S\}$ through only four numbers, namely, $\underline{\theta}^s_\mu(z_0), \overline{\theta}^s_\mu(z_0)$, $P^s(S)$ and $C^s(S)$. Thanks to previous numerical work that uses the measures μ_k and the sets A_k (see (44) and (42)) as approximations of μ and S, respectively, we have estimates given by our algorithms P_k of $P^s(S)$ (see [22]) and C_k of $C^s(S)$ (see [23]) and precise error bounds for such estimates. We show in Theorem 31 below how to obtain estimates $\underline{\xi}_k$ of $\underline{\theta}^s_\mu(z_0)$ and $\overline{\xi}_k$ of $\overline{\theta}^s_\mu(z_0)$, that such estimates converge to the real values, and we give accurate bounds for them, that is $\underline{\theta}^s_\mu(z_0) \in [\underline{\xi}_k^{\inf}, \underline{\xi}_k^{\sup}]$ and $\overline{\theta}^s_\mu(z_0) \in [\overline{\xi}_k^{\inf}, \overline{\xi}_k^{\sup}]$ (see the definition of $\underline{\xi}_k$, $\overline{\xi}_k$ and of the intervals $[\underline{\xi}_k^{\inf}, \underline{\xi}_k^{\sup}]$ and $[\overline{\xi}_k^{\inf}, \overline{\xi}_k^{\sup}]$ in Theorem 31). This allows us to implement an algorithm along the lines of those developed for the estimation of $C^s(S)$ and $P^s(S)$ (see [22,23]).

Theorem 31 *For $k > 1$, let*

$$\underline{\xi}_k := \min\left\{\mathring{\theta}^s_{\mu_k}(z_0, d) : \ d = |x - z_0|\,,\ x \in A_k,\ d \in [\frac{1}{2} - 2^{-k}, 1]\right\} \tag{46}$$

and

$$\overline{\xi}_k := \max\left\{\theta^s_{\mu_k}(z_0, d) : \ d = |x - z_0|\,,\ x \in A_k,\ d \in [\frac{1}{2} - 2^{-k}, 1]\right\} \tag{47}$$

be the estimates of $\underline{\theta}^s_\mu(z_0)$ and $\overline{\theta}^s_\mu(z_0)$, respectively. Let d_k be such that $\mathring{\theta}^s_{\mu_k}(z_0, d_k) = \underline{\xi}_k$, and let D_k be such that $\theta^s_{\mu_k}(z_0, D_k) = \overline{\xi}_k$.

Then,

$$\{\underline{\theta}^s_\mu(z_0), \underline{\xi}_k\} \in [\underline{\xi}_k^{\inf}, \underline{\xi}_k^{\sup}], \tag{48}$$

and

$$\{\overline{\theta}^s_\mu(z_0), \overline{\xi}_k\} \in [\overline{\xi}^{\inf}_k, \overline{\xi}^{\sup}_k], \tag{49}$$

where

$$\underline{\xi}^{\inf}_k = \underline{K}_k \underline{\xi}_k, \quad \underline{K}_k = (1-2^{1-k})^s, \quad \underline{\xi}^{\sup}_k = \frac{\mu_k(\mathring{B}(z_0, d_k + 2^{-k}))}{(2d_k)^s}, \tag{50}$$

$$\overline{\xi}^{\inf}_k = \frac{\mu_k(B(z_0, D_k - 2^{-k}))}{(2D_k)^s}, \quad \overline{K}_k = (1+2^{1-k})^s, \quad \overline{\xi}^{\sup}_k = \overline{K}_k \overline{\xi}_k. \tag{51}$$

Proof. That $\underline{\xi}_k \in [\underline{\xi}^{\inf}_k, \underline{\xi}^{\sup}_k]$ and $\overline{\xi}_k \in [\overline{\xi}^{\inf}_k, \overline{\xi}^{\sup}_k]$ is obvious from the definitions.

We prove first that $\underline{\theta}^s_\mu(z_0) \in [\underline{\xi}^{\inf}_k, \underline{\xi}^{\sup}_k]$. Using Lemma 30 (i) and (39), we obtain

$$\underline{\theta}^s_\mu(z_0) \le \frac{\mu(B(z_0, d_k))}{(2d_k)^s} \le \frac{\mu_k(\mathring{B}(z_0, d_k + 2^{-k}))}{(2d_k)^s} = \underline{\xi}^{\sup}_k.$$

Let $d \in [\frac{1}{2}, 1]$ be such that $\underline{\theta}^s_\mu(z_0) = \frac{\mu(B(z_0,d))}{(2d)^s}$. Lemma 30 (ii) guarantees the existence of $y_k \in A_k$ such that $\mu(B(z_0, d)) \ge \mu_k(\mathring{B}(z_0, d_{y_k}))$, where $d_{y_k} := |y_k - z_0| \in [d - 2^{-k}, d + 2^{-k}] \subset [\frac{1}{2} - 2^{-k}, 1]$. This, together with (46) and the inequality $d \ge \frac{1}{2}$ gives

$$\begin{aligned}\underline{\theta}^s_\mu(z_0) &= \frac{\mu(B(z_0, d))}{(2d)^s} \ge \frac{\mu_k(\mathring{B}(z_0, d_{y_k}))}{(2d)^s} = \left(\frac{d_{y_k}}{d}\right)^s \frac{\mu_k(\mathring{B}(z_0, d_{y_k}))}{(2d_{y_k})^s} \\ &\ge \left(\frac{d_{y_k}}{d}\right)^s \underline{\xi}_k \ge \left(\frac{d-2^{-k}}{d}\right)^s \underline{\xi}_k \ge \underline{\xi}^{\inf}_k.\end{aligned}$$

The proof that $\overline{\theta}^s_\mu(z_0) \in [\overline{\xi}^{\inf}_k, \overline{\xi}^{\sup}_k]$ is analogous. Using Lemma 30 (i) and (40), we obtain

$$\overline{\theta}^s_\mu(z_0) \ge \frac{\mu(B(z_0, D_k))}{(2D_k)^s} \ge \frac{\mu_k(B(z_0, D_k - 2^{-k}))}{(2D_k)^s} = \overline{\xi}^{\inf}_k.$$

Let $D \in [\frac{1}{2}, 1]$ be such that $\overline{\theta}^s_\mu(z_0) = \frac{\mu(B(z_0,D))}{(2D)^s}$. Lemma 30 (ii) guarantees the existence of $z_k \in A_k$ such that $\mu(B(z_0, D)) \le \mu_k(B(z_0, d_{z_k}))$, where $d_{z_k} := |z_k - z_0| \in [D - 2^{-k}, D + 2^{-k}] \subset [\frac{1}{2} - 2^{-k}, 1]$. This, together with (47) and the inequality $D \ge \frac{1}{2}$ gives

$$\begin{aligned}\overline{\theta}^s_\mu(z_0) &= \frac{\mu(B(z_0, D))}{(2D)^s} \le \frac{\mu_k(B(z_0, d_{z_k}))}{(2D)^s} \\ &= \left(\frac{d_{z_k}}{D}\right)^s \frac{\mu_k(B(z_0, d_{z_k}))}{(2d_{z_k})^s} \le \left(\frac{d_{z_k}}{D}\right)^s \overline{\xi}_k \\ &\le \left(\frac{D+2^{-k}}{D}\right)^s \overline{\xi}_k \le \overline{\xi}^{\sup}_k.\end{aligned}$$

■

We present in Table 1 the estimates $\underline{\xi}_k$, and $\overline{\xi}_k$ of $\underline{\theta}^s_\mu(z_0)$ and $\overline{\theta}^s_\mu(z_0)$ (see (48) and (49) for definitions), respectively, and the corresponding lower and upper bounds in the 100% confidence intervals $[\underline{\xi}_k^{\inf}, \underline{\xi}_k^{\sup}]$, $[\overline{\xi}_k^{\inf}, \overline{\xi}_k^{\sup}]$ (see (51),(49)) obtained by our algorithm for $k = 14$ (see the definition these values in (46), (47), (50) and (51)). We also provide the radii, d_k and D_k, of the μ_k-optimal balls.

See in Fig. 2a the graph of the function $\theta^s_{\mu_{14}}(z_0, d)$ as a function of $d \in [\varepsilon, 1]$, and in Fig. 2b the points $(g(d), \theta^s_{\mu_{14}}(z_0, d))$, where $g(d) := \varepsilon + \frac{\varepsilon - 1}{\log(\varepsilon)}(\log(d) - \log(\varepsilon))$ and $\varepsilon = 0.05$. This is a suitable logarithmic scale, [11], which allows us to see the periodicity of this function at such a scale.

We present in Table 2 the estimates P_k of $P^s(S)$ and C_k of $C^s(S)$ obtained by our algorithms for $k = 14$. The lower and upper bounds of $P^s(S)$ are denoted by $P_k^{\inf}$ and $P_k^{\sup}$, respectively, and the bounds of $C^s(S)$ are denoted by $C_k^{\inf}$ and $C_k^{\sup}$, respectively. These results were computed in [22] and [23]), respectively. Recall that $(P^s(S))^{-1}$ and $(C^s(S))^{-1}$ are the μ-densities of the balls of minimum and maximum μ-density in the set of typical balls. The estimates P_k and C_k are obtained by replacing S with A_k and μ with μ_k. Again, we have used open balls in the estimation of the density of the ball of minimum μ_k-density, and closed balls for the density of the ball of maximum μ_k-density. The centre and radius of the open ball of minimum μ_k-density are denoted by x_k^* and d_k, respectively, and the centre and radius corresponding to the closed ball of maximum μ_k-density are denoted by y_k^* and D_k. The table also contains the corresponding optimal μ_k-densities and their bounds. The upper bound $P_k^{\sup} := K_k^P P_k$ of $P^s(S)$ is slightly improved here with respect to the one given in [22]. Here $K_k^P := (1 - \frac{2^{5-k}}{\sqrt{3}})^{-s}$ instead of the value $K_k^P = (1 - \frac{2^{6-k}}{\sqrt{3}})^{-s}$ used in [22]. This gives the value $P_{14}^{\sup} = 1.671292$ given in Table 2 instead of the value $P_{14}^{\sup} = 1.668305$ given in Table 1 in [22]. The results of the following corollary are based on the estimates of Tables 1 and 2.

Corollary 32 *Let S be the Sierpinski gasket.*

(i) For any $\alpha \in \mathcal{M}^s\lfloor_S$, $Spec(\alpha, S)$ is the union of two closed disjointed intervals.

(ii)

$$Spec(\mu, S) \sim [0.2997, 0.3567] \cup [0.5994, 0.9951]$$

$$[0.2998, 0.3566] \cup [0.5999, 0.9944] \subset Spec(\mu, S) \subset [0.2996, 0.3568] \cup [0.5983, 0.9970]$$

(iii)

$$Spec(P^s\lfloor_S, S) \sim [0.5, 0.5951] \cup [1, 1.6602]$$

$$[0.5010, 0.5945] \cup [1, 1.6578] \subset Spec(P^s\lfloor_S, S) \subset [0.4995, 0.5963] \cup [1, 1.6662]$$

(iv)

$$Spec(C^s\lfloor_S, S) \sim [0.3012, 0.3584] \cup [0.6023, 1]$$

$$[0.3015, 0.3577] \cup [0.6032, 1] \subset Spec(C^s\lfloor_S, S) \subset [0.3005, 0.3588] \cup [0.6002, 1]$$

Proof. We know (see (38) in Theorem 14) that

$$Spec(\mu, S) = \left[\underline{\theta}^s_\mu(z_0), \overline{\theta}^s_\mu(z_0)\right] \cup \left[\frac{1}{P^s(S)}, \frac{1}{C^s(S)}\right], \tag{52}$$

and that (see (19))

$$Spec(\alpha, S) = \alpha(S) Spec(\mu, S), \quad \alpha \in \mathcal{M}^s\lfloor_S. \tag{53}$$

The two intervals in $Spec(\alpha, S)$, $\alpha \in \mathcal{M}^s\lfloor_S$ are disjointed if $\overline{\theta}^s_\mu(z_0) < \frac{1}{P^s(S)}$. Such a condition holds (see Theorem 31, and Tables 1 and 2) because

$$\overline{\theta}^s_\mu(z_0) \leq \overline{\xi}^{\sup}_{14} < 0.3568$$

and

$$\frac{1}{P^s(S)} \geq \frac{1}{P^{\sup}_{14}} > 0.5983.$$

Using (52), Theorem 31 and (53), we have that

$$Spec(\mu, S) \sim [\underline{\xi}_{14}, \overline{\xi}_{14}] \cup \left[\frac{1}{P_{14}}, \frac{1}{C_{14}}\right],$$

$$\left[\underline{\xi}^{\sup}_{14}, \overline{\xi}^{\inf}_{14}\right] \cup \left[\frac{1}{P^{\inf}_{14}}, \frac{1}{C^{\sup}_{14}}\right] \subset Spec(\mu, S) \subset \left[\underline{\xi}^{\inf}_{14}, \overline{\xi}^{\sup}_{14}\right] \cup \left[\frac{1}{P^{\sup}_{14}}, \frac{1}{C^{\inf}_{14}}\right],$$

$$Spec(P^s\lfloor_S, S) \sim [P_{14}\underline{\xi}_{\mu_{14}}, P_{14}\overline{\xi}_{\mu_{14}}] \cup \left[1, \frac{P_{14}}{C_{14}}\right],$$

$$\left[P^{\sup}_{14}\underline{\xi}^{\sup}_{14}, P^{\inf}_{14}\overline{\xi}^{\inf}_{14}\right] \cup \left[1, \frac{P^{\inf}_{14}}{C^{\sup}_{14}}\right] \subset Spec(P^s\lfloor_S, S) \subset \left[P^{\inf}_{14}\underline{\xi}^{\inf}_{14}, P^{\sup}_{14}\overline{\xi}^{\sup}_{14}\right] \cup \left[1, \frac{P^{\sup}_{14}}{C^{\inf}_{14}}\right],$$

(and, analogously, for $Spec(C^s\lfloor_S, S)$), and the proof is completed using the corresponding estimates of Tables 1 and 2. ∎

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$\underline{\xi}_{14}$	$[\underline{\xi}_{14}^{\inf}, \underline{\xi}_{14}^{\sup}]$	d_{14}
0.299714	[0.299656,0.299763]	0.642272

$\overline{\xi}_{14}$	$[\overline{\xi}_{14}^{\inf}, \overline{\xi}_{14}^{\sup}]$	D_{14}
0.356687	[0.356645,0.356756]	0.913663

Table 1: Extreme densities at z_0

Estimates of $\underline{\theta}^s_\mu(z_0)$ and $\overline{\theta}^s_\mu(z_0)$, bounds and radii, d_k and D_k, of the μ_k-optimal balls for $k = 14$.

x^*_{14}	d_{14}	P_{14}	$[P_{14}^{\inf}, P_{14}^{\sup}]$	$(P_{14})^{-1} = \mathring{\theta}^s_{\mu_{14}}(x^*_{14}, d_{14})$	$\left[(P_{14}^{\sup})^{-1}, (P_{14}^{\inf})^{-1}\right]$
(0.5,0)	0.160543	1.668305	[1.667178, 1.671292]	0.599411	[0.598339, 0.599816]
y^*_{14}	D_{14}	C_{14}	$[C_{14}^{\inf}, C_{14}^{\sup}]$	$(C_{14})^{-1} = \theta^s_{\mu_{14}}(y^*_{14}, D_{14})$	$\left[(C_{14}^{\sup})^{-1}, (C_{14}^{\inf})^{-1}\right]$
$\left(\frac{5}{16}, \frac{\sqrt{3}}{16}\right)$	0.145957	1.004903	[1.003109,1.005611]	0.995121	[0.994420, 0.996901]

Table 2: Packing and Centred measure estimates of S

Centres and radii of the balls $\mathring{B}(x^*_{14}, d_{14})$ and $B(y^*_{14}, D_{14})$ of minimum and maximum μ_{14}-densities, estimates P_{14} and C_{14} of $P^s(S)$ and $C^s(S)$, and bounds. The last two columns in the table are the μ_{14}-densities of the optimal balls (inverses of $P^s(S)$ and $C^s(S)$) and their bounds.

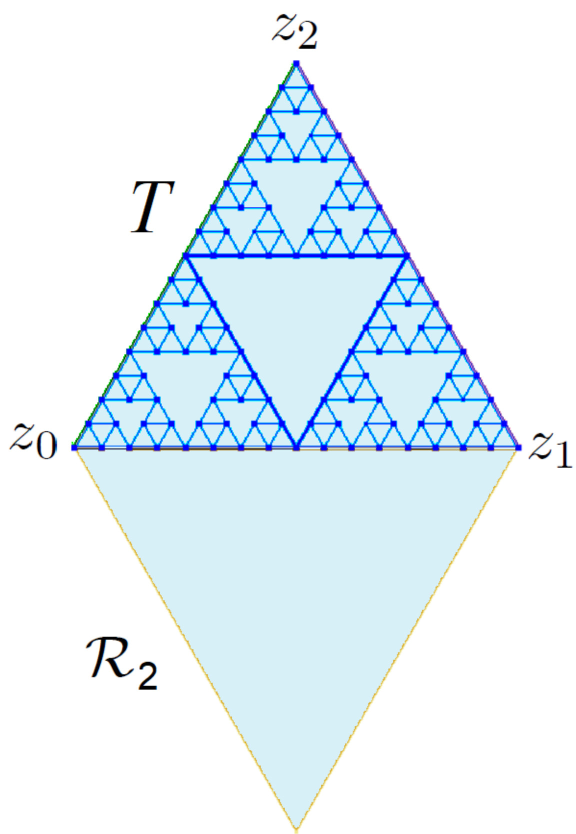
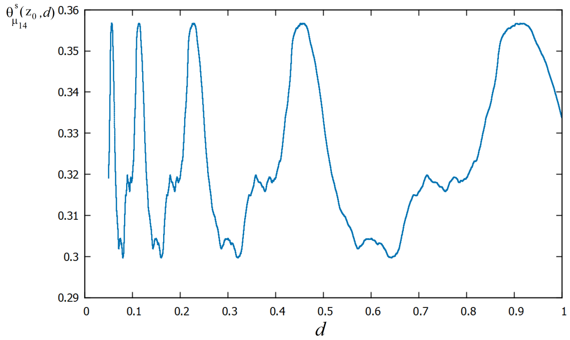
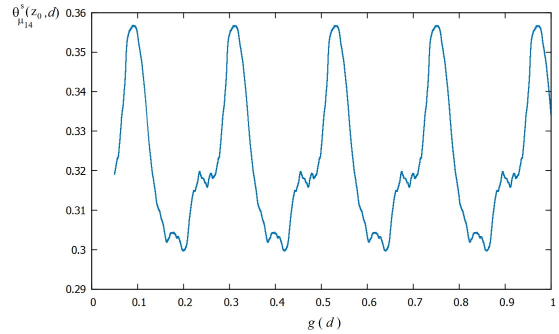


Figure 1: A feasible open set.

An open rhombus $\mathcal{R}_2$ that is a feasible open set for S.



(a) Values of $\theta^s_{\mu_{14}}(z_0, d)$ for $d \in [\varepsilon, 1]$ and $\varepsilon = 0.05$.



(b) Values of $(g(d), \theta^s_{\mu_{14}}(z_0, d))$, where $g(d) := \varepsilon + \frac{\varepsilon - 1}{\log(\varepsilon)}(\log(d) - \log(\varepsilon))$ and $\varepsilon = 0.05$.

Figure 2: Densities at z_0