On the Centred Hausdorff Measure of the Sierpiński Gasket

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Abstract

We show that the centred Hausdorff measure, $C^s(S)$, with $s = \frac{\log 3}{\log 2}$, of the Sierpiński gasket S, is C-computable (continuous-computable), in the sense that its value is the solution of the minimisation problem of a continuous function on a compact domain. We also show that $C^s(S)$ is A-computable (algorithmic-computable) in the sense that there is an algorithm that converges to $C^s(S)$, with explicit error bounds. Using this algorithm we show that $C^s(S) \sim 1.0049$.

Keywords: Self-Similar Sets, Sierpiński Gasket, Hausdorff Measures, Density of Measures, Computability of Fractal Measures.

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1 Introduction and main results

The role played by the Sierpiński gasket in the beginnings of Fractal Geometry might well be compared to that played by the triangle or the circle in the start of Euclidean Geometry. In spite of this fact some basic parameters of the Sierpiński gasket remain unknown. The complex nature of fractal objects has given rise to a great variety of measurement tools such as metric measures and dimensions in Euclidean spaces. The Hausdorff measure, $\mathcal{H}^s$, the centred Hausdorff measure, C^s, the spherical Hausdorff measure, $\mathcal{H}^s_{sph}$, and the packing measure, P^s, (see Sec. 1.2 for definitions) are some of the existing ones that are so necessary to capture the multiple geometric properties of fractal objects. In these measures, which we shall refer to as s-measures, the exponent s is a real positive number. We shall denote by $\mathcal{M}^s$ the set $\left\{\mathcal{H}^s, C^s, P^s, \mathcal{H}^s_{sph}\right\}$ of metric s-measures.

For integer values of s all metric s-measures are multiples of the corresponding Lebesgue measure $\mathcal{L}^s$, so that they are well understood and computable for some families of conspicuous subsets.

If s is not an integer, the examples of s-sets (i.e. sets with finite and positive s-measure), for which the exact value of any of these measures is known, may be considered exceptional even in the class of self-similar sets with strong separation condition properties (see, for example, [1–7] and the references therein). In fact, the exact value of the mentioned s-measures of almost any self-similar set, including the Sierpiński triangle or the Koch curve, is not known.

The nature of the results shown in this paper for the centred Hausdorff measure (see also [8] for an example with the packing measure) indicates that finding the exact value of a metric measure of a self-similar set is essentially a problem of a computational nature. As a rule of thumb, the required computations are easier when the constituent parts of the corresponding set are suitably separated, but as this separation becomes shorter, the study of their properties becomes computationally more arduous (see [9]).

In this paper we undertake the issue of computing the centred Hausdorff measure of the Sierpiński gasket, S, a self-similar set for which the separation among its constituent parts is zero, but it satisfies the open set condition (OSC for the sequel, see Sec. 1.1 below). To the best of our knowledge, this is the first known computation of the centred Hausdorff measure of a connected self-similar set with OSC. In Theorem 7 we show that $C^s(S)$, with $s = \frac{\log 3}{\log 2}$, is C-computable, in the sense that its value is the solution of a minimisation problem on a compact domain. See [1] for an example of a self-similar set K in the line, with OSC, for which the Hausdorff measure is not C-computable. In Theorem 10 we show that $C^s(S)$ is A-computable, providing an algorithm whose output converges to $C^s(S)$, with error bounds tending to zero.

We are witnessing a flourishing of applications of fractal geometry, its tools and methods, to solve complex problems from different fields (see some examples in [10–13]). In keeping with the complexity of the underlying problems, the application of some of these tools requires the construction of computational algorithms, non-existent to date, and this is where the contribution of this article should be placed.

We shall first summarise some basic definitions and notation to understand the problem.

1.1 The Sierpiński gasket

The Sierpiński gasket or Sierpiński triangle (see Fig. 1) is a special case of a self-similar set which is generated by the iterated function system (IFS), $\Psi = \{f_0, f_1, f_2\}$, of three contracting similitudes of the plane, with contraction ratios $r_i := \frac{1}{2}$, $i \in M := \{0, 1, 2\}$, given by

$$f_i(x) = \frac{1}{2}x + v_i,\ i \in M \quad \text{where } v_0 = (0,0),\ v_1 = (\frac{1}{2}, 0),\ \text{and } v_2 = \frac{1}{4}(1, \sqrt{3}).$$

We use composite indices, $i := i_1, i_2, ..., i_k \in M^k$, to denote the compositions $f_i := f_{i_1} \circ f_{i_2} \circ ... \circ f_{i_k}$ and we write r_i for the contraction ratio of f_i (which equals 2^{-k} if $i \in M^k$). We shall denote as z_i to the fixed point of each f_i, $i \in M$, that is, $z_i = 2v_i$, $i \in M$.

The Sierpiński gasket S, as the attractor of Ψ, is the invariant set of the Hutchinson operator, F, defined, for $A \subset \mathbb{R}^2$, by

$$F(A) := f_0(A) \cup f_1(A) \cup f_2(A), \tag{1}$$

S being the unique non-empty compact set admitting the self-similar decomposition

$$S = f_0(S) \cup f_1(S) \cup f_2(S) = F(S).$$

S can be parameterised as $S = \{\pi(i) : i \in \Sigma\}$ with parameter space $\Sigma := M^\infty$ and *geometric projection mapping* $\pi : \Sigma \to S$ given by $\pi(i) = \cap_{k=1}^{\infty} f_{i(k)}(S)$, where $i(k) \in M^k$ denotes the k-th curtailment $i_1 \dots i_k$ of $i = i_1 i_2 \cdots \in \Sigma$. Notice that π is non-injective. We adopt the convention

$M^0 = \emptyset$ and write $M^* = \cup_{k=0}^{\infty} M^k$ for the set of words of finite length. For any $i \in M^*$, the cylinder sets are denoted by $S_i := f_i(S)$, and $S_i := S$, if $i \in M^0$. For $i \in M^k$, S_i is a *cylinder of the* k-th generation, or k-cylinder.

Notation 1 *Throughout the document the equilateral triangle whose vertices are z_i, $i \in M$ will be denoted by T, and we shall abbreviate $T_i := f_i(T)$ if $i \in M^*$, $k \in \mathbb{N}^+$, and $T_i := T$ if $i \in M^0$ (see Fig. 2).*

The system of similitudes Ψ satisfies the OSC (see [14]) meaning that there is an open set $\mathcal{R} \subset \mathbb{R}^2$ satisfying $f_i(\mathcal{R}) \subset \mathcal{R}$ for all $i \in M$ and $f_i(\mathcal{R}) \cap f_j(\mathcal{R}) = \varnothing$ for $i,\, j \in M$, $i \neq j$. We will refer to such a set $\mathcal{R}$ as *a feasible open set* (for S). Furthermore, if $\mathcal{R} \cap S \neq \varnothing$ then $\mathcal{R}$ satisfies the strong open set condition SOSC (cf. [15–17]). One feasible open set that fulfils the SOSC is the open rhombus $\mathcal{R}$ composed of the topological interior of the union of T and its reflection across the edge of T opposite the point z_2 (see Fig. 2). As Ψ satisfies the OSC, the dimension of all metric measures are the same (see [18, 19] for the notion of dimension of a measure), and they also coincide with the similarity dimension, $\dim S = \frac{\log 3}{\log 2}$, which is the value that satisfies $\sum_{i=0}^{2} r_i^{\dim S} = 1$.

The Sierpiński triangle S can also be considered in terms of a probability measure supported by the set S. Let $\mathcal{P}(\mathbb{R}^2)$ be the space of compactly supported probability Borel measures on $\mathbb{R}^2$ and let $\mathbf{M} : \mathcal{P}(\mathbb{R}^2) \to \mathcal{P}(\mathbb{R}^2)$ be the Markov operator, defined by

$$\mathbf{M}(\alpha) = \sum_{i=0}^{2} \frac{1}{3} \left(\alpha \circ f_i^{-1}\right),\ \alpha \in \mathcal{P}(\mathbb{R}^2).$$

The operator $\mathbf{M}$ is contractive on $\mathcal{P}(\mathbb{R}^2)$, equipped with a suitable metric (see [20, 21]). Its unique fixed point, μ^*, is called the *invariant* or, sometimes, *natural probability measure.* It is supported on S and satisfies

$$\mathbf{M}^k(\alpha) = 3^{-k} \sum_{i \in M^k} \left(\alpha \circ f_i^{-1}\right) \xrightarrow[k\to\infty]{w} \mu^* \tag{2}$$

for any $\alpha \in \mathcal{P}(\mathbb{R}^2)$. Here, $\xrightarrow{w}$, denotes the weak convergence of measures and $\mathbf{M}^k = \mathbf{M} \circ \mathbf{M} \circ ... \circ \mathbf{M}$ is the k-th iterate of $\mathbf{M}$. Furthermore, μ^* coincides with the projection on S of ν,

$$\mu^* = \nu \circ \pi^{-1}, \tag{3}$$

where ν is the Bernoulli measure on Σ that gives weight $\frac{1}{3}$ to each symbol in M (see [20]).

By (3), we know that $\mu^*(S_i) = 3^{-k}$ for $i \in M^k$. On the other hand, any metric measure $\alpha \in \mathcal{M}^{\dim S}$ scales under similitudes, i.e. $\alpha(S_i) = r_i^{\dim S}\alpha(S) = 3^{-k}\alpha(S)$, for $i \in M^k$. Since μ^* and α, are multiples on cylinder sets, they are indeed multiple measures, and then, all the normalised measures $(\alpha(S))^{-1}\,\alpha\lfloor_S$ coincide with μ^* and with the normalised Hausdorff measure $\mu := \frac{\mathcal{H}^s\lfloor_S}{\mathcal{H}^s(S)}$, with $s = \dim S$. Here, $\beta\lfloor_S$ stands for the restriction of the measure β to S.

Remark 2 *From now on we shall work with the measure μ rather than with other metric measures on S, bearing in mind that if $s = \dim S$, A is a Borel set and $\alpha \in \mathcal{M}^s\lfloor S$ with $\mathcal{M}^s\lfloor S := \left\{\mathcal{H}^s\lfloor_S, C^s\lfloor_S, P^s\lfloor_S, \mathcal{H}^s_{sph}\lfloor_S\right\}$, we have*

$$\alpha(A) = \alpha(S)\mu(A). \tag{4}$$

Whence the computation of $\alpha(A)$ boils down to the computation of $\alpha(S)$ plus the computation of $\mu(A)$.

Notation 3 *From now on, we shall write $\mathcal{B}$ for the set of closed balls $B(x,d)$ centred at $x \in S$ and with radius $d > 0$ satisfying that there is some feasible open set $\mathcal{R}$ for S with $B(x,d) \subset \mathcal{R}$.*

1.2 Metric measures and their relationship with s-densities

In this paper we refer to the set $\mathcal{M}^s = \left\{\mathcal{H}^s, C^s, P^s, \mathcal{H}^s_{sph}\right\}$ as the set of metric measures (cf. Chapter 1 in [22]).

Following Saint Raymond and Tricot (see [23]), the Hausdorff centred measure, $C^s(A)$, of a subset $A \subset \mathbb{R}^n$ is defined in a two-step process (see [24] for definitions on general metric spaces). First, the premeasure $C_0^s(A)$ is defined for any $s > 0$ by

$$C_0^s(A) = \lim_{\delta\to 0} \inf\left\{\sum_{i=1}^{\infty}(2d_i)^s \ : \ 2d_i \le \delta,\ i = 1, 2, \dots\right\}, \tag{5}$$

where the infimum is taken over all coverings, $\{B(x_i, d_i)\}_{i\in\mathbb{N}^+}$, of A by closed balls $B(x_i, d_i)$ centred at points $x_i \in A$. The second step, needed by the lack of monotonicity of $C_0^s(A)$ due to the restriction $x_i \in A$, $i \in \mathbb{N}^+$ (cf. [25] and [26, Example 4]), leads to the following definition of the *centred Hausdorff s-dimensional measure*

$$C^s(A) = \sup\left\{C_0^s(F) : \ F \subset A,\ F \text{ closed}\right\}.$$

However, in [26] it is proved that, if E is a self-similar set with OSC, $A \subset \mathbb{R}^n$ is a compact set, and s is its similarity dimension [20], then $C^s(A \cap E) = C_0^s(A \cap E)$ implying that the second step can be omitted. As we shall see, this makes it possible to reduce the problem of calculating this fractal measure to the computation of the optima of certain density functions (see (6)).

In regard to metric measures based on packings, the standard packing measure, P^s, is relevant in this research. It is defined in a two-step process, first by taking

$$P_0^s(A) = \lim_{\delta\to 0} \sup \left\{ \sum_{i=1}^{\infty} (2d_i)^s : \quad 2d_i \leq \delta,\ i = 1, 2, \ldots \right\},$$

where the supremum above is taken over all δ-*packings* $\{B(x_i, d_i)\}_{i\in\mathbb{N}^+}$, with $x_i \in A$ for all i, and $B(x_i, d_i) \cap B(x_j, d_j) = \varnothing$ for $i \neq j$, and then

$$P^s(A) = \inf \left\{ \sum_{i=1}^{\infty} P_0^s(F_i) \right\},$$

where the infimum is taken over all coverings $\{F_i\}_{i\in\mathbb{N}^+}$ of A by closed sets F_i. When A is a compact set with $P_0^s(A) < \infty$, then $P^s(A) = P_0^s(A)$ holds, and the second step above might be omitted (see [27]).

The *spherical s-dimensional Hausdorff measure,* $\mathcal{H}^s_{sph}(A)$, is obtained by removing in (5) the requirement that the balls are centred at points of A. Finally, the classical *s-dimensional Hausdorff measure,* $\mathcal{H}^s(A)$, results if coverings of A by arbitrary subsets, $\{U_i\}_{i\in\mathbb{N}^+}$, are considered and $2d_i$ is replaced in (5) with the diameter of U_i, $|U_i|$. No second step is required for these two last measures.

Let us recall that, for $\alpha \in \cup_{s>0}\mathcal{M}^s$ and $A \subset \mathbb{R}^n$, the α-dimension of A is defined by $\dim_\alpha A = \inf\{s : \alpha(A) = 0\}$ and that $\dim_\alpha(S) = \frac{\log 3}{\log 2}$ (see [19] for the definition of the dimension of a set with respect to a metric measure).

We now introduce the s-densities, a useful tool for analysing the behaviour of metric measures defined for measures $\alpha \in \mathcal{M}^s\lfloor_A$ by

$$\theta^s_\alpha(x, d) = \frac{\alpha(B(x, d))}{(2d)^s},\ s > 0,\ x \in \mathbb{R}^n,\ d > 0. \tag{6}$$

A fundamental result in geometric measure theory (Marstrand's theorem, [28]) states that, except when s is an integer, the limiting values

$$\overline{\theta}^s_\alpha(x) = \limsup_{d\to 0} \theta^s_\alpha(x, d) \text{ and } \underline{\theta}^s_\alpha(x) = \liminf_{d\to 0} \theta^s_\alpha(x, d), \tag{7}$$

called *upper and lower* s-densities of α respectively, cannot coincide on subsets $A \subset \mathbb{R}^n$ of positive α-measure. Furthermore, classical results in fractal geometry (see, for instance, [29]) show that these limiting behaviours on subsets $A \subset \mathbb{R}^n$ of positive α-measure can determine global properties of these subsets. In particular, there are bounds on the values $\alpha(A)$, $A \subset \mathbb{R}^n$, $\alpha \in \{\mathcal{H}^s\lfloor_A,\ P^s\lfloor_A\}$ provided that the values $\overline{\theta}^s_\alpha(x)$ and $\underline{\theta}^s_\alpha(x)$ are bounded for $x \in A$. These deep results are quite general, as they stand for arbitrary subsets of $\mathbb{R}^n$. Unfortunately, they are useless for computing $\alpha(A)$ for general A, since the computation of the involved limits is out of reach. But if E is a self-similar set satisfying OSC and $s = \dim E$, then Morán proved in [30] that,

$$P^s(E) = \sup\Big\{ \left(\theta^s_\mu(x,d)\right)^{-1} :\ B(x,d) \in \mathcal{B}\Big\}, \tag{8}$$

(recall Notation 3). In that reference, analogous results for $\mathcal{H}^s(E)$ and $\mathcal{H}^s_{sph}(E)$ are also proved and they were extended in [26] to the characterisation of $C^s(E)$, namely

$$C^s(E) = \inf\Big\{ \left(\theta^s_\mu(x,d)\right)^{-1} :\ x \in E \ \text{ and } \ d > 0\Big\}. \tag{9}$$

However, even with these results, the numerical computation of the values defined in (8) and (9) is still out of reach without further restrictions on the set of balls, since the computational time grows exponentially as their diameters decrease (see discussion in [9]).

In [9,31–33], it was shown that, for self-similar sets E where the *strong separation condition* (SSC) holds (i.e. $f_i(E) \cap f_j(E) = \varnothing$ for $i, j \in M$, $i \neq j$), the determination of $\alpha(E)$, $\alpha \in \{C^s\lfloor_E, P^s\lfloor_E\}$, starts to be computationally amenable, as the classes of balls to be explored can be reduced to those centred at E and with diameters within a known interval bounded away from zero. As the function $(\theta^s_\alpha(x,d))^{-1}$ is known to be a continuous function, both in x and in d (see [19]), the supremum and infimum in (8) and (9) then became a maximum and a minimum, respectively. In the terminology of this article, they are C-computable. The method is not only able to render estimates of $P^s(E)$ and $C^s(E)$, but also allows an explicit construction of optimal coverings and packings [31]. Moreover, explicit formulas for $\alpha(E)$ can be found under additional, stronger forms of separation than SSC (see [34]).

1.3 Results

In this paper we make computational work with the Sierpiński gasket S, where SSC does not hold, and the results mentioned above cannot be applied. From now on we set $s = \dim S = \frac{\log 3}{\log 2}$. Using the symmetries of S, Theorem 7 in Sec. 2, narrows down the search in the class of balls given in (9) to the class of closed balls centred at points of the cylinder set of the second generation S_{01} (which amounts, from a computational viewpoint, to $\frac{1}{9}$ of the points in the discrete approximations of S). Furthermore, the existence of internal homotheties permits the suppression of balls with a radius outside the interval $\left[\frac{\sqrt{3}}{16}, \frac{\sqrt{3}}{8}\right]$, leading to a C-computable problem that is easier, in general, than the SSC case if the distance among the 1-cylinders is smaller than $\frac{\sqrt{3}}{16}$. The issue of computation of exact values of metric measures in self-similar sets with OSC other than S remains, so far, a challenge.

From a theoretical point of view, the results in Theorem 7 show that S is a C^s-exact self-similar set (see [35]), which means that there exist a ball for which the maximal density is attained, and an optimal covering by balls centred on S for which the exact value of $C^s(S)$ is reached.

In Sec. 3, we outline the main ideas for tackling the computational task through an algorithm aimed at approaching the value of $C^s(S)$, with which we obtain the following estimate

$$C^s(S) \sim 1.0049.$$

In Theorem 10 we give, in addition to the estimates of $C^s(S)$, explicit lower and upper bounds for $C^s(S)$ provided by the algorithm at each stage k, and we show that the errors tend to zero (A-computability of $C^s(S)$).

Since $\theta^s_\alpha(x,d) = \alpha(S)\theta^s_\mu(x,d)$, for $\alpha \in \mathcal{M}^s\lfloor_S$, if we gather the estimate and bounds for $C^s(S)$ with the corresponding to $P^s(S)$ given in [8] then, from the full range of values of $\theta^s_\mu(x,d)$ on balls in $\mathcal{B}$ (recall Notation 3), we obtain the information of the full range of values of $\theta^s_\alpha(x,d)$ for $\alpha \in \{\mu, C^s\lfloor_S, P^s\lfloor_S\}$. See in Fig. 3 the balls that our algorithm give as balls of maximum and minimum density, respectively.

If we consider the definitions of $C^s(S)$ and $P^s(S)$, their values capture global covering and packing properties of S. We now see that they also convey essential information about the geometry of the family of spherical neighbourhoods of S.

By (9) we know that, for any $x \in S$ any $d > 0$,

$$\frac{\mu(B(x,d))}{(2d)^s} \le C^s(S)^{-1} \tag{10}$$

holds, and that this uniform upper bound for $\theta^s_\mu(x,d)$ is tight for the whole family of spherical neighbourhoods, $\{B(x,d),\ x \in S,\ d > 0\}$ (i.e. it is the least common upper bound for that family of balls).

Analogously we can obtain a uniform lower bound for $\theta^s_\mu(x,d)$ through (8). Let A_1 be the set of the three fixed points of Ψ. It can be easily seen [35] that if $x \in S - A_1$ and d is small enough, then

$B(x,d)\in\mathcal{B}$, so (8) gives

$$\frac{\mu(B(x,d))}{(2d)^s}\geq P^s(S)^{-1}. \tag{11}$$

Thus, $P^s(S)^{-1}$ yields a uniform lower bound for $\theta^s_\mu(x,d)$, which is tight for the family of balls $\mathcal{B}$.

Furthermore, the values of $C^s(S)$ and $P^s(S)$ also give an essential information about the pointwise behavior of the densities $\theta^s_\mu(x,d)$ for μ-a.e. $x\in S$. In order to study such behaviour, we consider the spectrum of asymptotic α-densities of subsets A of S, which is defined as

$$Spec(\alpha,A)=\left\{\lim_{n\to\infty}\theta^s_\alpha(x,r_n):x\in A,\ \ \lim_{n\to\infty}r_n=0\right\},$$

for $\alpha\in\left\{\mu, C^s\lfloor_S, P^s\lfloor_S, \mathcal{H}^s\lfloor_S, \mathcal{H}^s_{sph}\lfloor_S\right\}$. In [35] it is proved that

$$Spec(\mu,x)=\left[\underline{\theta}^s_\mu(x),\overline{\theta}^s_\mu(x)\right]=\left[\frac{1}{P^s(S)},\frac{1}{C^s(S)}\right], \tag{12}$$

for μ-a.e. $x\in S$ (see notation in (7)). An analogous result is true for any self-similar set E satisfying SOSC). Thus, $Spec(\mu,x)$ is an interval which gives the full range of variation of the asymptotic densities at μ-a.e. $x\in S$, range that, by (12), is determined by the values of $C^s(S)$ and $P^s(S)$. Note that (12) implies that the bounds in (10) and (11) are tight, μ-a.e. x, for the family of balls $\{B(x,r),\ x\in E\}$.

More in general, it is known (see [29]) that if a self-similar set E of dimension s is rectifiable, then s must be an integer and $P^s(E)=C^s(E)$, so the length of $Spec(\mu,x)$ is null for μ-*a.e.* $x\in E$. In contrast, if E is unrectifiable (as it happens to occur, for instance, if s is non-integer), then $P^s(E)>C^s(E)$ and the length of $Spec(\mu,x)$ which, by [35], is known to be constant for μ-*a.e.* $x\in E$, gives an useful index of the irregularity of E.

Using the estimates and bounds for $C^s(S)$ given in Sec. 3, together with the corresponding ones for $P^s(S)$ given in [8], we get that $Spec(\mu,x)\sim[0.5994,0.9951]$, that

$$Spec(\mu,x)\subset[0.5983,0.9970] \tag{13}$$

μ-a.e. $x\in S$ and we also get $Spec(\alpha,x)$ for $\alpha\in\{C^s\lfloor_S,P^s\lfloor_S\}$ (see [35] for the full spectrum $Spec(\alpha,S)$, $\alpha\in\{\mu,C^s\lfloor_S,P^s\lfloor_S\}$). These are the first known non-trivial estimates for the spectrum of asymptotic densities for metric s-dimensional measures on self-similar sets with OSC.

Further, using the bounds

$$0.77\leq\mathcal{H}^s(S)\leq 0.8192$$

obtained algorithmically by P. Mora in [36] from a theoretical result of B. Jia in [37], and (13), we are able to produce the first non-trivial estimate for the asympotic spectrum of $\mathcal{H}^s$ on self-similar sets with OSC, namely

$$Spec(\mathcal{H}^s\lfloor_S, x) = \mathcal{H}^s(S)Spec(\mu, x) \subset [0.4607, 0.8167],$$

for $\mathcal{H}^s$-a.e. $x \in S$.

2 Computability of the centred Hausdorff measure of S

In this section we prove the C-computability (Theorem 7) and the A-computability (Theorem 10) of $C^s(S)$.

2.1 C-computability of $C^s(S)$

The symmetry of the Sierpiński gasket can be leveraged to achieve a crucial reduction on the set of the candidate balls to be optimal given in (9), conducive to handling the computability problem of $C^s(S)$ with a suitable algorithm. The following two properties, valid for a general set $E \subset \mathbb{R}^n$, are useful.

Lemma 4 *Let $E \subset \mathbb{R}^n$, $\alpha \in \mathcal{M}^s\lfloor_E$ and $h : \mathbb{R}^n \to \mathbb{R}^n$ be a similarity with scaling factor r_h.*

(i) If $A \subset \mathbb{R}^n$ satisfies $h(A \cap E) = h(A) \cap E$, then $\alpha(h(A)) = r_h^s \alpha(A)$.

(ii) If $C \subset E$ satisfies $h(C) \subset E$, then

$$\alpha\Big(B(h(x), r_h d) \cap h(C)\Big) = r_h^s \alpha(B(x, d) \cap C).$$

Proof. It is well known that all metric measures scale under similarities so that, if $\beta \in \mathcal{M}^s$, h is a similarity with contraction ratio r_h and $A \subset \mathbb{R}^n$, then $\beta(h(A)) = r_h^s \beta(A)$. Under our hypothesis

$$\beta\lfloor_E(h(A)) = \beta\left(h(A) \cap E\right) = \beta(h\left(A \cap E\right)) = r_h^s \beta\lfloor_E(A).$$

This proves (i). Under the hypotheses in (ii), taking $B(x, d) \cap C$ as A in (i)

$$\begin{aligned} h\Big(B(x,d) \cap C \cap E\Big) &= h\Big(B(x,d) \cap C\Big) = h(B(x,d)) \cap h(C) \\ &= h(B(x,d)) \cap h(C) \cap E = h\Big(B(x,d) \cap C\Big) \cap E, \end{aligned}$$

and (ii) follows. ■

Remark 5 *If we take $E = S$ in Lemma 4, then α can be replaced with μ, since μ is a multiple of the measures in $\mathcal{M}^s\lfloor_S$. In general, if E is a self-similar set with OSC and $s = \dim E$, then α can be taken to be the invariant measure of E (for the Markov operator).*

The proof of Theorem 7 below relies upon finding a reduced set of balls where all the relevant values of $\theta^s_\mu(x,d)$ are attained. A basic tool to this end is the notion of density equivalent balls.

Definition 6 *We say that $B(x,d)$ with $x \in S$ is density equivalent to $B(x',d')$, if $x' \in S$ and*

$$\theta^s_\mu(x,d) = \theta^s_\mu(x',d').$$

Theorem 7

$$C^s(S) = \min\left\{ \left(\theta^s_\mu(x,d)\right)^{-1} : \quad x \in S_{01},\ \frac{\sqrt{3}}{16} \le d \le \frac{\sqrt{3}}{8}. \right\} \tag{14}$$

Proof. We show first that

$$C^s(S) = \inf\left\{ \left(\theta^s_\mu(x,d)\right)^{-1} : \quad x \in S_{01},\ d > 0 \right\}. \tag{15}$$

Let π_i be the reflection across the altitude, h_i, of T (recall Notation 1) through z_i, $i \in M$ (see Fig. 2). Since $\pi_1(S_2) = S_0$, $\pi_2(S_1) = S_0$ and $\pi_0(S_{02}) = S_{01}$, Lemma 4 (i) applied to π_i and μ (see also Remark 5), implies, on the one hand, that any ball centred in S_i, $i = 1,2$ is density equivalent to a ball centred in S_0 and, on the other hand, that any ball centred in S_{02} is density equivalent to a ball centred in S_{01}. So, we can restrict our search to balls centred in $S_{00} \cup S_{01}$.

Let $x \in S_{00}$ and $d > 0$. In order to show that we can neglect $B(x,d)$, consider first the case $B(x,d) \cap S \subset S_0$. Then Lemma 4(ii) implies that $B(x,d)$ is density equivalent to $B(f_0^{-1}(x), 2d)$. Further, if $f_0^{-1}(x) \in S_{02}$, then $B(x,d)$ is density equivalent to $B(\pi_0(f_0^{-1}(x)), 2d)$, centred in S_{01}. If $f_0^{-1}(x) \in S_{00}$, we can repeat the argument k times until $f_0^{-k}(x) \in S_{02} \cup S_{01}$ or $B(f_0^{-k}(x), 2^k d) \cap (S_1 \cup S_2) \neq \emptyset$.

Consider now the case $x \in S_{00}$ and $B(x,d) \cap (S_1 \cup S_2) \neq \emptyset$. Then, $B(x,d)$ is either not optimal or it is density-equivalent to some ball centred in S_{01} because

$$\left(\theta^s_\mu(x,d)\right)^{-1} = \frac{(2d)^s}{\mu(B(x,d))} \ge \frac{(2d)^s}{\mu(B(\pi_4(x),d))} = \left(\theta^s_\mu(\pi_4(x),d)\right)^{-1}, \tag{16}$$

where π_4 is the reflection across the altitude h_4 of T_0 through $f_2(z_0)$ (see Fig. 2). The inequality in (16) follows from

$$\mu(B(x,d)) \le \mu(B(\pi_4(x),d)). \tag{17}$$

In order to check (17), we decompose $B(x, d)$ into the union

$$B(x, d) \cap S = \Big(B(x, d) \cap S_0\Big) \cup \Big(B(x, d) \cap (S_1 \cup S_2)\Big) \tag{18}$$

and analogously,

$$B(\pi_4(x), d) \cap S = \Big(B(\pi_4(x), d) \cap S_0\Big) \cup \Big(B(\pi_4(x), d) \cap (S_1 \cup S_2)\Big). \tag{19}$$

Since $\pi_4(S_0) = S_0$, Lemma 4(ii) gives

$$\mu\left(B(x, d) \cap S_0\right) = \mu\left(B(\pi_4(x), d) \cap S_0\right).$$

Now, if $y \in B(x, d) \cap (S_1 \cup S_2)$, since $S_1 \cup S_2$ is contained in the right hand half plane determined by h_4 and x belongs to the left-hand half plane, we see that $|\pi_4(x) - y| \le |x - y|$ holds, and

$$B(x, d) \cap (S_1 \cup S_2) \subset B(\pi_4(x), d) \cap (S_1 \cup S_2). \tag{20}$$

Thus, in the decompositions given in (18) and (19) the μ-measure of the first terms are equal, whilst, by (20), the μ-measure of the second term cannot be smaller in (19) than in (18), which gives (17).

Once we have proved that (15) holds, we show that we can restrict the search to balls with radii within the range $[\frac{\sqrt{3}}{16}, \frac{\sqrt{3}}{8}]$.

Let $x \in S_{01}$ and $d < \frac{\sqrt{3}}{16}$. Clearly, $B(x, d) \cap S \subset S_0 \cup S_1$.

Suppose first that $B(x, d) \cap S \subset S_0$, then we can apply Lemma 4 (ii) to conclude that $B(f_0^{-1}(x), 2d)$, centred in S_1, is density equivalent to $B(x, d)$. Moreover, we have already seen that such a ball is either density equivalent to a ball of equal or greater radius and centred in S_{01} or cannot be optimal. Hence, we can iterate the argument till $d \ge \frac{\sqrt{3}}{16}$ or we find a density equivalent ball centred in S_{01} and intersecting S_1. Observe that if we need k iterations of the argument to achieve a ball of radius $2^k d \ge \frac{\sqrt{3}}{16}$, then $2^{k-1} d < \frac{\sqrt{3}}{16} \le 2^k d$, implying that $2^k d \in [\frac{\sqrt{3}}{16}, \frac{\sqrt{3}}{8}]$.

Now, if $B(x, d) \cap S_1 \ne \emptyset$ (recall that we have assumed that $d < \frac{\sqrt{3}}{16}$), then $x \in S_{011}$ and $B(x, d) \cap S = B(x, d) \cap (S_{01} \cup S_{100})$. Therefore, if we take the homothety, $h : \mathbb{R}^2 \to \mathbb{R}^2$, of ratio 2 and fixed point at the unique point of $S_0 \cap S_1$, $h(S_{01}) = S_0$ and $h(S_{100}) = S_{10}$ imply that

$$\begin{aligned} h\Big(B(x, d) \cap S\Big) &= h\Big(B(x, d) \cap (S_{01} \cup S_{100})\Big) = B(h(x), 2d) \cap (S_0 \cup S_{10}) \\ &= B(h(x), 2d) \cap S. \end{aligned}$$

Lemma 4(i) shows then that $\mu(B(h(x),2d)) = 2^s\mu(B(x,d))$, which implies that $B(h(x),2d)$ is density equivalent to $B(x,d)$. If $\frac{\sqrt{3}}{16} \leq 2d$, we have concluded because $2d \in [\frac{\sqrt{3}}{16}, \frac{\sqrt{3}}{8}]$ and $h(x) \in S_{01}$. Otherwise, we can repeat the argument as many times as needed until $\frac{\sqrt{3}}{16} \leq 2d$.

Finally, if $x \in S_{01}$ and $d > \frac{\sqrt{3}}{8}$, then, since $f_1(B(x,d) \cap S) = B(f_1(x), \frac{d}{2}) \cap S_1$ and $f_1(x) \in S_{101}$ we get, by Lemma 4 (ii), that

$$\begin{aligned}\mu(B(x,d)) = \mu(B(x,d) \cap S) = 2^s\mu\left(B\left(f_1(x), \frac{d}{2}\right) \cap S_1\right)\\ \leq 2^s\mu\left(B\left(f_1(x), \frac{d}{2}\right)\right) = 2^s\mu\left(B\left(\pi_2(f_1(x)), \frac{d}{2}\right)\right),\end{aligned} \tag{21}$$

with $\pi_2(f_1(x)) \in S_{01}$.

The proof of (14) concludes by noticing that (21) implies that $(\theta^s_\mu(x,d))^{-1} \geq (\theta^s_\mu(\pi_2(f_1(x)), \frac{d}{2}))^{-1}$ and that we can repeat this procedure $k \in \mathbb{N}^+$ times until $\frac{\sqrt{3}}{16} \leq 2^{-k}d \leq \frac{\sqrt{3}}{8}$, obtaining on each step a ball centred in S_{01} with less or equal inverse density. ■

2.2 A-computability of $C^s(S)$

In this section we construct a discrete algorithm that converges to $C^s(S)$ and we provide error bounds tending to zero for its estimates, thus showing that $C^s(S)$ is A-computable.

Following the structure of the algorithms developed in [9], [26] and [32], the construction of such an algorithm relies upon the relationship between the centred Hausdorff measure and the inverse density function given in Theorem 7. With the aim of finding a computationally adequate estimate of the minimum value given in (14), a discrete approximation of both the Sierpiński gasket and the invariant measure is proposed.

Firstly, it is well-known that, for any non-empty compact subset $A \subset \mathbb{R}^2$, S can be built with an arbitrary level of detail by increasing the iterations k in $F^k(A)$, where $F^k = F \circ F ... \circ F$ is the k-th iterate of the Hutchinson operator F (see (1)). This is because $\lim_{k\to\infty} F^k(A) = S$ with respect to the Hausdorff metric, given that S is the attractor of Ψ under the contracting operator F (cf. [20]). Furthermore, if $A \subset S$, then $F^k(A) \subset S$ for any $k \in \mathbb{N}^+$. We use these facts in the design of our algorithm where we take as initial compact set $A_1 := \{z_0, z_1, z_2\}$ (recall that $z_i \in M$ are the fixed points of the similitudes in Ψ) and obtain the set

$$A_k := F^{k-1}(A_1) \subset S,\ k \geq 2 \tag{22}$$

as a discrete approximation of S, at iteration k, of our algorithm.

If we take $\alpha = \mu_1 := \frac{1}{3}(\delta_{z_0}+\delta_{z_1}+\delta_{z_2})$ as the initial measure in (2), where δ_x is the Dirac probability measure at x, then

$$\mu_k := \mathbf{M}^{k-1}(\mu_1) = \frac{1}{3^{k-1}} \sum_{i\in M^{k-1}} \mu_1 \circ f_i^{-1} = \frac{1}{3^k} \sum_{i\in M^{k-1}} \left(\delta_{f_i(z_0)} + \delta_{f_i(z_1)} + \delta_{f_i(z_2)}\right) \tag{23}$$

is a probability measure supported on $A_k \subset S$ and $\mu_k \xrightarrow[k\to\infty]{w} \mu$.

The discrete measure μ_k is the approximation of the invariant measure μ that our algorithms take at iteration k.

For $i \in M^{k-1}$, $\delta_{f_{i_1\ldots i_{k-2}}(z_{i_{k-1}})}$ is an atom of μ_k and a summand in the right-hand term in (23). Since $f_i(z_j) = f_j(z_i)$, for i, $j \in M$, with $i \neq j$, all the points in $A_2 - A_1$ have two codes in M^2. From this, it easily follows that all the points in $A_k - A_1$ also have two codes in M^k, $k \geq 2$, (see [8], Sec. 3), and therefore we can write (23) as

$$\mu_k = \frac{1}{3^k}(\delta_{z_0} + \delta_{z_1} + \delta_{z_2}) + \frac{2}{3^k} \sum_{x\in A_k\setminus A_1} \delta_x. \tag{24}$$

The algorithm outlined in Sec. 3 works with the sets $A_k \subset S$ defined in (22) and with the measures μ_k defined in (24) as approximations of the Sierpiński gasket S and of the invariant measure μ at iteration k, respectively.

Addressing the issue of establishing error bounds in the estimates of the μ-measure of balls through the approximating μ_k-measures, the following lemma allows us to make the comparison of the μ measure and the μ_k-measure in two relevant cases.

Lemma 8

(i) Let $\left\{S_i : i \in I \subset M^k\right\}$, $k \in \mathbb{N}^+$, be a collection of k-cylinder sets. Then

$$\mu\left(\bigcup_{i\in I} S_i\right) \leq \mu_k\left(\bigcup_{i\in I} S_i\right)$$

(ii) Let $A \subset S$, $k \in \mathbb{N}^+$, and $I := \left\{i \in M^k : S_i \cap A\right\} \neq \varnothing$. Then

$$\mu_k(A) \leq \mu\left(\bigcup_{i\in I} S_i\right)$$

Proof. For any $i \in I$ we know that $\mu(S_i) = 3^{-k}$, so

$$\mu\left(\bigcup_{i\in I} S_i\right) = \sum_{i\in I} \mu\left(S_i\right) = 3^{-k}\#(I) \tag{25}$$

where $\#(I)$ denotes the cardinality of I. We know that each cylinder S_i contains a unique point, $\hat{x} \in A_k$ (namely $\hat{x} := \pi(i_1 i_2 ... i_{k-1} i_k i_k i_k ...)) = f_{i_1 i_2 ... i_{k-1}}(z_{i_k})$), and that each point of $(\bigcup_{i \in I} S_i) \cap A_k$ either belongs to a unique k-cylinder in $\{S_i : i \in I\}$ (let us write $A_{k,1}(I)$ for such subset of A_k), or belongs to the set $A_{k,2}(I)$, or subset of points of A_k that lie on two k-cylinders of $\{S_i : i \in I\}$. Hence

$$A_k \cap \left(\bigcup_{i \in I} S_i \right) = A_{k,1}(I) \cup A_{k,2}(I)$$

with $A_{k,1}(I) \cap A_{k,2}(I) = \varnothing$. Then, since

$$A_1 \cap \left(\bigcup_{i \in I} S_i \right) \subset A_{k,1}(I) \cap \left(\bigcup_{i \in I} S_i \right)$$

and, in consequence $A_1 \cap A_{k,2}(I) = \varnothing$, we have

$$\begin{aligned} \mu_k \left(\bigcup_{i \in I} S_i \right) &= \mu_k \left(\left(\bigcup_{i \in I} S_i \right) \cap A_k \right) \\ &= \mu_k \Big(A_{k,1}(I) \cap A_1 \Big) + \mu_k \Big(A_{k,1}(I) \cap (A_k - A_1) \Big) + \mu_k \Big(A_{k,2}(I) \cap (A_k - A_1) \Big) \\ &= \frac{1}{3^k} \left\{ \# \Big(A_{k,1}(I) \cap A_1 \Big) + 2\# \Big(A_{k,1}(I) \cap (A_k - A_1) \Big) + 2\# \Big(A_{k,2}(I) \cap (A_k - A_1) \Big) \right\} \\ &\geq \frac{1}{3^k} \left\{ \# \left(A_{k,1}(I) \right) + 2\# \Big(A_{k,2}(I) \cap (A_k - A_1) \Big) \right\} = \frac{1}{3^k} \left\{ \# \left(A_{k,1}(I) \right) + 2\# \left(A_{k,2}(I) \right) \right\}. \end{aligned} \tag{26}$$

Let $J := \{i \in M^k : i = i_1 i_1 ... i_1, \ i_1 \in M\}$, $k > 1$ and $i \in M^k \backslash J$. For each cylinder S_i, there is $i^* \in M^k \backslash J$ such that $S_i \cap S_{i^*} = S_i \cap S_{i^*} \cap A_k$ consists of a unique point. Consider the partition of I, $I = I_0 \cup I_1 \cup I_2$ where $I_0 := J \cap I$, $I_1 := \{i \in I : \ \ i^* \notin I\}$ and $I_2 := \{i \in I : \ \ i^* \in I\}$. Since there is a bijective mapping between $A_{k,1}(I)$ and $I_0 \cup I_1$, an injective mapping between $A_{k,2}(I)$ and I_2, and each cylinder S_i, $i \in M^k$ contains a unique point in $A_{k,2}(I)$, we get that $\#(A_{k,1}(I)) = \#(I_0) + (\#(I_1))$ and $2\#(A_{k,2}(I)) = (\#(I_2))$. This, together with (26) and (25) gives

$$\mu_k \left(\bigcup_{i \in I} S_i \right) \geq \frac{1}{3^k} \# (I) = \mu \left(\bigcup_{i \in I} S_i \right).$$

Observe that the equality holds if and only if the set in the other term affected by a coefficient 2, $A_{k,1}(I) \cap (A_k - A_1)$, is empty.

In order to prove (ii), note that $A_{k,1}(I) \cap (A_k - A_1) = \varnothing$, since if some cylinder in $\{S_i : i \in I\}$ intersects the set $A \cap (A_k - A_1)$ at a point x, then there is another cylinder to which x also belongs and this cylinder also belongs in turn to the collection of cylinders $\{S_i : i \in I\}$. Thus, using the above decomposition of $\mu_k\left(\left(\bigcup_{i\in I} S_i\right)\cap A_k\right)$ and by the final observation,

$$\begin{aligned}\mu_k(A) = \mu_k(A\cap A_k) &\le \mu_k\left(\left(\bigcup_{i\in I} S_i\right)\cap A_k\right)\\ &= \mu_k\Big(A_{k,1}(I)\cap A_1\Big) + \mu_k\Big(A_{k,2}(I)\cap (A_k - A_1)\Big)\\ &= \frac{1}{3^k}\#\left(I\right) = \mu\Big(\bigcup_{i\in I} S_i\Big).\end{aligned}$$

■

Theorem 10 below establishes the A-computability of $C^s(S)$. It is a discrete version of Theorem 7, suitable for our computational purposes as it gives the estimates and the lower and upper bounds of $C^s(S)$ at each iteration k, namely, C_k, $C_k^{\inf}$ and $C_k^{\sup}$ respectively. We first prove the following lemma.

Lemma 9 *Let $k > 0$, $x \in \mathbb{R}^2$ and $d > 2^{-k}$. Then,*

(i) $\mu(B(x,d)) \ge \mu_k(B(x,d-2^{-k}))$

(ii) If $S \nsubseteq B(x,d)$ and $B(x,d)\cap A_k \ne \varnothing$, then there is $y_k \in A_k$ satisfying $\mu(B(x,d)) \le \mu_k(B(x,d_k))$, where $d_k = |y_k - x|$ and $d - 2^{-k} \le d_k \le d + 2^{-k}$.

Proof. Let $k > 0$, $x \in \mathbb{R}^2$ and $d > 2^{-k}$.

(i) Let

$$J_k := \{i \in M^k : S_i \subset B(x,d)\}$$

and

$$H_k := \{i \in M^k : B(x,d-2^{-k})\cap S_i \ne \emptyset\}.$$

Clearly, $H_k \subset J_k$ holds. Then, using Lemma 8 (ii)

$$\mu_k(B(x,d-2^{-k})) \le \sum_{i\in H_k}\mu(S_i) \le \sum_{i\in J_k}\mu(S_i) \le \mu(B(x,d)).$$

(ii) We prove first that, if $S \nsubseteq B(x,d)$ and $B(x,d)\cap A_k \ne \varnothing$, the set

$$G_k := \{i \in M^k : \partial B(x,d)\cap S_i \ne \emptyset\}$$

is non empty, where $\partial B(x,d)$ is the border of $B(x,d)$. Let $U(x,d)$ denote the open ball centred at x and with radius d. If $G_k = \varnothing$, then

$$\bigcup_{i\in M^k} S_i \subset U(x,d)\cup\left(B(x,d)\right)^c. \qquad (27)$$

Since $B(x,d) \cap A_k \neq \varnothing$ and we may assume that $\partial B(x,d) \cap A_k = \varnothing$ or G_k would be trivially nonempty, we see that $U(x,d) \cap (\cup_{i \in M^k} S_i) \neq \varnothing$, and $S \nsubseteq B(x,d)$ implies that the set $(B(x,d))^c \cap (\cup_{i \in M^k} S_i)$ is nonempty, so (27) contradicts that for any $k > 0$, $\cup_{i \in M^k} S_i$ is a connected set and our claim that $G_k \neq \varnothing$ is proved.

Let

$$I_k := \{i \in M^k : B(x,d) \cap S_i \neq \emptyset\}$$

and

$$d_k := \max\{|y - x| : \ y \in A_k \cap (\cup_{i \in I_k} S_i)\}.$$

Let $z \in A_k$ satisfying $d_k = |z - x|$. Then, $d_k \leq d + 2^{-k}$, since $\cup_{i \in I_k} S_i \subset B(x, d + 2^{-k})$ and, using part (i) of Lemma 8, we have

$$\begin{aligned} \mu(B(x,d)) &\leq \mu(\cup_{i \in I_k} S_i) \leq \mu_k(\cup_{i \in I_k} S_i) \\ &= \mu_k\left(\left(\cup_{i \in I_k} S_i\right) \cap A_k\right) \leq \mu_k\left(B(x, d_k)\right). \end{aligned}$$

Finally, using that $\partial B(x,d) \cap (\cup_{i \in M^k} S_i) \neq \varnothing$ we see that

$$d_k \geq \max\{|y - x| : \ y \in A_k \cap (\cup_{i \in G_k} S_i)\} \geq d - 2^{-k}.$$

■

Theorem 10 *Let*

$$C_k = \min\left\{(\theta^s_{\mu_k}(x,d))^{-1} : \ x \in A_k \cap S_{01} \ \textit{and} \ d = |y - x| \ \textit{with} \ y \in A_k \ \textit{and} \ \frac{\sqrt{3}}{16} \leq d \leq \frac{\sqrt{3}}{8} + 2^{1-k}\right\}. \tag{28}$$

Then, for every $k \geq 4$,

$$C_k^{\inf} \leq C^s(S) \leq C_k^{\sup} \tag{29}$$

holds, where

$$C_k^{\inf} = K_k C_k, \quad K_k = \left(1 + \frac{2^{5-k}}{\sqrt{3}}\right)^{-s}, \quad C_k^{\sup} = \frac{(2d_k)^s}{\mu_k(B(x_k, d_k - 2^{-k}))} \tag{30}$$

and $B(x_k, d_k)$ is a ball minimising (28).

Proof. Let $k \geq 4$, and let $B(x_k, d_k)$ be a ball minimising (28). We know that $C_k = \left(\theta^s_{\mu_k}(x_k, d_k)\right)^{-1}$, and $\frac{\sqrt{3}}{16} \leq d_k \leq \frac{\sqrt{3}}{8} + 2^{1-k}$. Using (9) and Lemma 9 (i), we get

$$C^s(S) \leq \frac{(2d_k)^s}{\mu(B(x_k, d_k))} \leq \frac{(2d_k)^s}{\mu_k(B(x_k, d_k - 2^{-k}))} = C_k^{\sup},$$

where $C_k^{\sup}$ is well defined since $d_k - 2^{-k} \geq \frac{\sqrt{3}}{16} - 2^{-k} \geq 0$ and $\mu_k(B(x_k, d_k - 2^{-k})) > 0$, as $x_k \in B(x_k, d_k - 2^{-k}) \cap A_k$.

In order to prove the inequality $C_k^{\inf} \leq C^s(S)$, let $(x, d) \in S \times [\frac{\sqrt{3}}{16}, \frac{\sqrt{3}}{8}]$ be such that $C^s(S) = \frac{(2d)^s}{\mu(B(x,d))}$. Let $i \in M^k$ be such that $x \in S_i$, and let $y_k = S_i \cap A_k$. Since $B(x, d) \subset B(y_k, d + 2^{-k})$ we get

$$\mu(B(x, d)) \leq \mu(B(y_k, d + 2^{-k})). \tag{31}$$

Taking the ball $B(y_k, d + 2^{-k})$ as $B(x, d)$ in Lemma 9 (ii) we can get a point $z_k \in A_k$ satisfying that

$$\mu(B(y_k, d + 2^{-k})) \leq \mu_k(B(y_k, d_k^*)), \tag{32}$$

where $d_k^* := |z_k - y_k|$ and $d_k^* \in [d, d + 2^{1-k}] \subset [\frac{\sqrt{3}}{16}, \frac{\sqrt{3}}{8} + 2^{1-k}]$. Then using (31), (32), (28), that $d_k^* \leq d + 2^{1-k}$, and that $d \geq \frac{\sqrt{3}}{16}$ we get

$$\begin{aligned} C^s(S) &= \frac{(2d)^s}{\mu(B(x,d))} \geq \frac{(2d)^s}{\mu_k(B(y_k, d_k^*))} = \left(\frac{d_k^*}{d}\right)^{-s} \left(\theta^s_{\mu_k}(y_k, d_k^*)\right)^{-1} \\ &\geq \left(\frac{d_k^*}{d}\right)^{-s} C_k \geq \left(\frac{d + 2^{1-k}}{d}\right)^{-s} C_k \geq \left(1 + \frac{2^{5-k}}{\sqrt{3}}\right)^{-s} C_k = C_k^{\inf}. \end{aligned}$$

■

Remark 11 *Notice that the convergence of the algorithm follows directly from the convergence of* C_k *to* $C^s(S)$ *as* k *tends to infinity. The last holds true from (29), (30) and that the fact that* $C_k^{\inf} = K_k \frac{(2d_k)^s}{\mu_k(B(x_k, d_k))}$*, where* K_k *tends exponentially to* 1 *when* k *tends to infinity. The resulting upper bound for the error is* $C_k^{\sup} - C_k^{\inf}$*. Obtaining a desirable explicit bound in terms of* k *is, however, non-trivial as it depends on the number of* k*-cylinders intersected by the circular crown* $B(x_k, d_k) - B(x_k, d_k - 2^{-k})$ *(see [9] for explicit bounds depending on* k *for the errors of the estimations of* $C^s(E)$ *and* $P^s(E)$ *for self-similar sets* E *satisfying SSC).*

Theorem 10 gives error bounds in the estimation of $C^s(S)$ *in terms of the density of the ball selected by the algorithm as optimal, but it does not give any error bound on the estimation of the centre and of the radius of the optimal ball. Such estimations remain so far an open problem.*

Remark 12 *Recall (see Remark 2) that having an estimate of* $\alpha(S)$*, allows one to estimate* $\alpha(A)$ *for any Borel set* A*, and* $\alpha \in \mathcal{M}^s \lfloor S$*. In particular, taking* $\alpha = C^s \lfloor_S$ *and* $A = B(x, d)$ *in (4) we get*

$$C^s \lfloor_S (B(x, d)) = C^s(S) \mu(B(x, d)). \tag{33}$$

Then, for sufficiently large k, we have

$$C^s\lfloor_S(B(x,d)) \sim C_k\mu_k(B(x,d)), \tag{34}$$

where $\sim$ denotes approximate equality. Moreover, if $k \geq 4$, and $d > 2^{-k}$, then the combination of (33), (34), Theorem 10 and Lemma 9 guarantee

$$C_k^{\inf}\mu_k(B(x,d_1)) \leq C^s\lfloor_S(B(x,d)) \leq C_k^{\sup}\mu_k(B(x,d_2)), \tag{35}$$

where $d_1 := d - 2^{-k} > 0$, and $d_2 := d + 2^{-k}$.

3 Algorithm and numerical results

The numerical results of this section are obtained through an algorithm that computes C_k and the bounds of $C^s(S)$, $C_k^{\inf}$ and $C_k^{\sup}$, given in Theorem 10.

3.1 The algorithm

The structure of this new algorithm is akin to the one presented in [8] for $P^s(S)$, so we will simply describe it in general terms, making a comparison with its equivalent for the packing measure and referring the interested reader to [8] for further details.

The analogous result to Theorem 10 for $P^s(S)$ is included in the remark below (Theorem 7 in [8]). The proof of Theorem 10 using part (ii) of Lemma 9 introduces a novel approach that can be generalised to improve the values of d_0, K_k^P, and the restriction $k \geq 6$ in the remark below, which could be replaced with $\widetilde{d}_0 := \frac{\sqrt{3}}{16} - 2^{1-k}$, $\widetilde{K}_k^P := \left(1 - \frac{2^{5-k}}{\sqrt{3}}\right)^{-s}$, and $k \geq 4$ respectively. Remark 13 will ease the understanding of the changes required for the adaptation of the algorithm for the estimation of $P^s(S)$ to the estimation of $C^s(S)$.

Remark 13 *For every $k \geq 6$,*

$$P_k^{\inf} \leq P^s(S) \leq P_k^{\sup}$$

where

$$P_k := \max\left\{ \left(\mathring{\theta}^s_{\mu_k}(x,d)\right)^{-1} : \ x \in A_k \cap S_{01}, \ d = |y-x| \ \text{with } y \in A_k \backslash S_2 \ \text{and } d_0 \leq d \leq d_x \right\}, \tag{36}$$

$d_0 := \frac{\sqrt{3}}{16} - 2^{2-k}$, $d_x := \max\{|y-x| : y \in \partial\mathcal{R}\}$, *$\mathcal{R}$ is a feasible open set for S, $\mathring{\theta}^s_{\mu_k}(x,d)$ is the μ_k-density of the open ball $U(x,d)$,*

$$P_k^{\inf} := \frac{(2d_k)^s}{\mu_k(U(x_k, d_k + 2^{-k}))}, \quad K_k^P := \left(1 - \frac{2^{6-k}}{\sqrt{3}}\right)^{-s}, \quad P_k^{\sup} := K_k^P P_k, \tag{37}$$

and $U(x_k, d_k)$ is an open ball that maximises (36).

Let us briefly list the changes required to adapt the algorithm described in [8] and based on the result of Remark 13 to our case, avoiding unnecessary duplication.

1. **Replace maximums with minimums.** Recall that for any $k \geq 4$, the aim of the algorithm for the estimation of $C^s(S)$ is to find a ball, $B(x,d)$ of minimal inverse μ_k-density where $d = |x-y|$, $x \in A_k \cap S_{01}$, and $y \in A_k$. For the estimation of $P^s(S)$ the goal was to maximise the inverse density, so this has to be changed accordingly.

2. **New bounds.** The definitions of $C_k^{\inf}$, $C_k^{\sup}$ and K_k given in (30), are analogous to the ones of $P_k^{\inf}$, $P_k^{\sup}$ and K_k^P given in (37). Thus, to rewrite the algorithm the roles of $P_k^{\inf}$, K_k^P and $P_k^{\sup}$ have to be switched with those of $C_k^{\inf}$, K_k and $C_k^{\sup}$, respectively.

3. **Range of radii.** Computing the terms $\mu_k(B(x,d))$ and $(2d)^s$ needed to find the minimum value given in (28), requires the calculation of the distances from each $x \in A_k \cap S_{01}$ to all the points in A_k and the selection of those within the allowed range of radii given in (28). The μ_k-measure of the resulting balls is then obtained by arranging in a list the sequence of feasible distances in increasing order so that the position of a distance d in the list, together with the distances that are equal to d, provides the number of points of A_k within $B(x,d)$, and hence it gives $\mu_k(B(x,d))$. Since the constrains on the radii of these candidates to optimal balls differ from one case to another, it should be adapted by replacing $d_0 := \frac{\sqrt{3}}{16} - 2^{2-k}$ with $\frac{\sqrt{3}}{16}$ and d_x with $\frac{\sqrt{3}}{8} + 2^{1-k}$.

4. **Closed balls.** The bounds in Theorem 10, and in particular the inequality (32), require that the balls considered by the algorithm be closed, instead of open like those used in the computation of $P^s(S)$. Notice that since $\mu(\partial B(x,d)) = 0$ (see [19]), using open or closed balls in (14) does not make a difference, and therefore neither does it make a difference in (28) or (36) if k is large

enough. Only for small k do the results vary substantially. The use of closed balls implies that, in the number of points that contribute to the μ_k-measure of the ball $B(x,d)$, we have to consider the number of points, t_x, in $U(x,d)$, and the number of points, T_x, in $\partial B(x,d)$. Therefore we have to replace $\mu_k(U(x,d)) = \frac{2}{3^k} t_x$ in [8] with $\mu_k(B(x,d)) = \frac{2}{3^k}(t_x + T_x)$.

3.2 Numerical results

Table 1 shows the algorithm's output from the fifth to the fourteenth iteration. The output of the algorithm for $k = 14$ together with Theorem 10 gives the estimate $C_{14} = 1.004903$ of $C^s(S)$, and a 100% confidence interval for $C^s(S)$, $[C_{14}^{\text{inf}}, C_{14}^{\text{sup}}] = [1.003109, 1.005611]$, with a length of less than 0.002502. Moreover, for every iteration k the selected ball, $B(x_k, d_k)$, has the same centre $x_k := f_{010}(z_2) = (\frac{5}{16}, \frac{\sqrt{3}}{16})$ and its radius, d_k, varies slightly for any k in the range $k = 11, ..., 14$. The stability observed indicates that the inverse μ_k-density of the ball $B(x_k, 0.146)$ (the red colour ball in Fig. 3) results in a good approximation to the minimum value in (14).

Regarding the stability of the error bounds for $C^s(S)$, Table 1 shows that there are two fixed decimal places in the values of the last three iterations of C_k^{sup}, and a slightly slower stabilisation of C_k^{inf}. This is mainly due to the initially slow convergence to one of the terms K_k, which was also the reason why, in the algorithm for $P^s(S)$, the convergence of P_k^{sup} was slower than that of P_k^{inf} (see Sec. 4.3 in [8]). Since the values of C_k change at a slower rate from $k = 10$ on, and the lower and upper bounds of C_k are arbitrarily close to C_k for large enough k, we can conjecture that $C^s(S) \sim C_{14} \sim 1.0049$.

More precise estimates for $C^s(S)$ would require a refinement of the lower bound for $C^s(S)$ or a significant increment of the largest value taken by k, now fixed at $k_{\max} := 14$. Regarding computational time, the estimation of C_k, for any k, $k \le k_{max} \in \{10, 11, 12, 13, 14\}$ requires 5 seconds, 46 seconds, 7 minutes, 67 minutes and 9.7 hours, respectively. It is worth noting that for $k = 14$ the diameter of the k-th cylinders is small ($2^{-14} \sim 6 \times 10^{-5}$) and the cardinality of the set of k-th cylinders is large ($3^{14} = 4782969$). This makes computationally challenging to significantly increase k_{max}. Additionally, the algorithm used is not optimised for reducing the computation time, and it has run in a standard personal computer. Better results can be obtained in a HPC (high-performance computing) system with an optimised version of this algorithm.

If we gather these results with the estimate given in [8] for $P^s(S)$, we obtain a quite complete information of the total range of values of $\theta^s_\alpha(x,d)$, $\alpha \in \{C^s\lfloor_S, P^s\lfloor_S\}$.

In [8] what is obtained is the estimate $P_{15} = 1.6683$ for $P^s(S)$ at iteration $k = 15$, and balls $B(z_k, d_k)$ maximising (36) for $k \in \{6, ..., 15\}$ were found. For $k = 14, 15$ the centres of these balls are both the same $z_k := (0.5, 0)$, and their radii are $d_k \sim 0.1605$. Notice that the ball $B(z_k, 0.1605)$ is μ-density equivalent to a ball centred at $f_{010}(z_2)$, which is precisely the centre of the ball which minimises (28), and with radius $d \sim 0.08$. This ball is outlined in green in Fig. 3, together with the red colour ball whose inverse density gives the estimate of $C^s(S)$ at iteration $k = 14$.

The evolution of the inverse density $(\theta^s_{\mu_{14}}(f_{010}(z_2), d))^{-1}$ as a function of the radius, d, is plotted in Fig. 4. Notice that the minimum and maximum values of this function correspond to the approximations $C^s(S) \sim 1.0049$ and $P^s(S) \sim 1.6683$, respectively.

For a detailed discussion of the frontiers of the computation of metric measures see the section of conclusions in [9].

4 Conclusions

The centred Hausdorff measure, C^s, and the packing measure, P^s, form a dual pair in terms of which several local and global properties of the spherical neighborhoods of a metric space can be efficiently expressed (see [25], [24]). The computation of the particular values of $C^s(S)$ achieved in this paper, together with the computation of that of the packing measure, $P^s(S)$ in [8], complete a first step in this direction, with relevant information (see Sec. 1.3) on the bounds of the spherical densities, $\alpha(B(x,d))(2d)^{-s}$ and their range of asymptotic variation, $Spec(\alpha, x)$ for $\alpha \in \{\mu, C^s\lfloor_S, P^s\lfloor_S\}$.

A similar program can be developed for self-similar sets with SSC (see [31] and [9]). We have also been able to develop this programme for the Sierpiński gasket, with does not satisfy SSC but it satisfies OSC. In fact, our method makes the computational problem easier than that posed by self-similar sets which satisfy SSC with small distances between their primary cylinders.

This is achieved through a careful geometric analysis of the Sierpiński triangle and the strategic utilisation of its symmetries. Can the same method be applied to other self-similar sets in this situation?

In [38] we prove that the same procedure can be applied to the penta-Sierpiński gasket P. The method used there consists of a two-step process: In the search of balls of optimal density, we first reduce the set of their centers and discard balls of large radius by using symmetries of P and then we discard balls of small radius using internal homotheties. In [38] it is argumented how it seems likely that the method can be applied to the whole family of IFS constructed through homotheties with fixed points at the vertexes of regular polygons and with just touching 1-cylinders (primary cylinders with only two intersection points with the two adjacent 1-cylinders).

We also think that the method could be applied to fractals as the Koch curve. An anonymous referee pointed out that a natural family of self-similar sets with OSC to which our results could be extended is that of post-critically finite (p.c.f.) self-similar sets. This conjecture turns out to be true in the case of the penta-Sierpiński gasket P, and also, we think, in the case of the related family of regular fractal polygons described above and the Koch curve, but we do not know whether the same holds true for general p.c.f. self-similar sets.

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k	d_k	C_k^{inf}	C_k	C_k^{sup}
5	0.125	0.409736	0.843750	2.700000
6	0.143205	0.622414	0.930364	1.255991
7	0.143205	0.790389	0.978694	1.141810
8	0.144690	0.894667	0.999143	1.068851
9	0.144690	0.945925	1.000593	1.035149
10	0.147354	0.975686	1.003735	1.016677
11	0.145596	0.990358	1.004556	1.011856
12	0.145834	0.997550	1.004691	1.007754
13	0.145957	1.001285	1.004867	1.006332
14	0.145957	1.003109	1.004903	1.005611

Table 1: Estimates of the centred measure of S.

Algorithm outputs rounded to six decimal places: to the smallest value for C_k^{inf}, to the largest for C_k^{sup} and to the nearest for C_k and d_k.

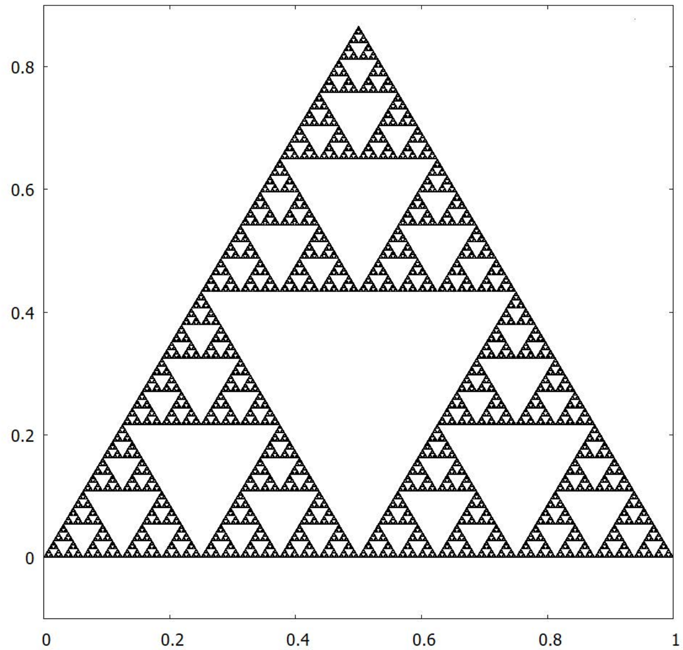

Figure 1: Sierpiński gasket S.

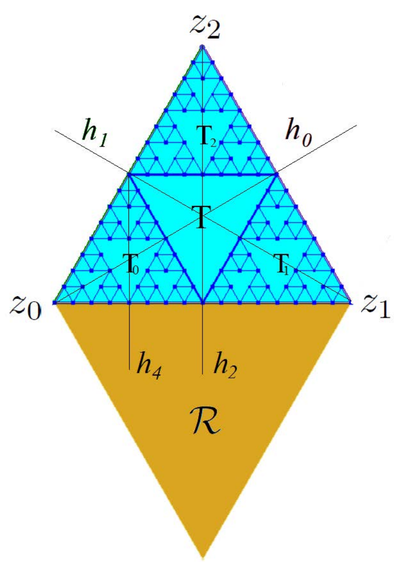

Figure 2: A feasible open set $\mathcal{R}$ for S.

The rhombus $\mathcal{R}$ is the topological interior of the union of T and its reflection across the edge opposite the point z_2. Triangles T and T_i, $i \in M$. Altitudes h_i of T through z_i, $i \in M$, and altitude h_4 of T_0 through $f_2(z_0)$.

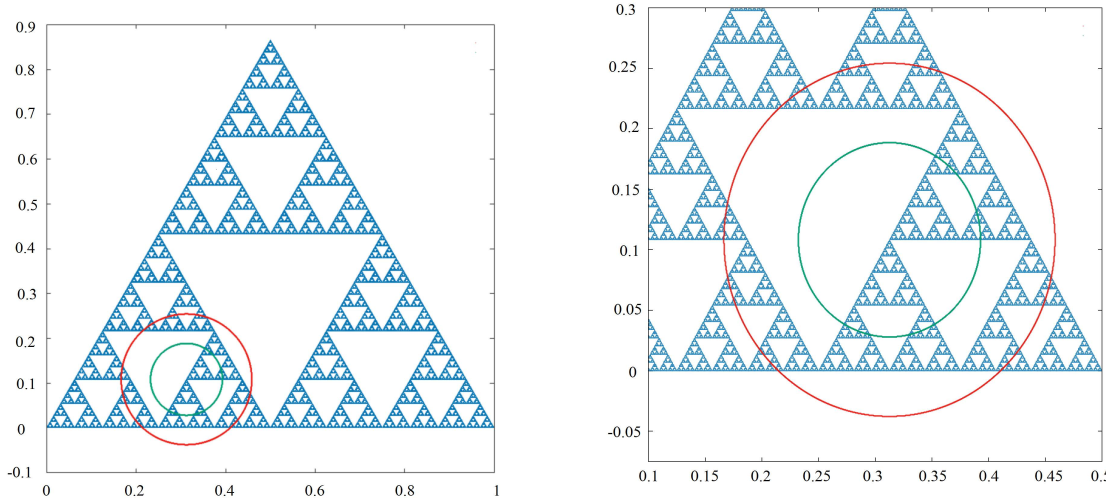

Figure 3: Optimal balls for the packing and centred measures of S.

Balls of minimum (in green) and maximum (in red) μ_{14}-density.

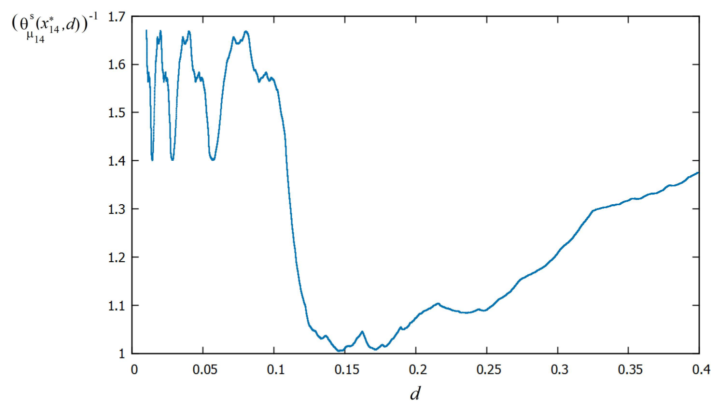

Figure 4: Inverse of the density at the optimal point.

Graph of $(\theta^s_{\mu_{14}}(x^*_{14}, d))^{-1}$ as a function of d with $x^*_{14} = f_{010}(z_2)$. The minimal value of $(\theta^s_{\mu_{14}}(x^*_{14}, d))^{-1}$ corresponds to $C^s(S) \sim 1.0049$ and the maximal one to $P^s(S) \sim 1.6683$.