Irregularity index and spherical densities of the Penta-Sierpinski gasket

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Abstract. We compute the centred Hausdorff measure, $C^s(\mathbf{P}) \sim 2.44$, and the packing measure, $P^s(\mathbf{P}) \sim 6.77$, of the penta-Sierpinski gasket, $\mathbf{P}$, with explicit error bounds. We also compute the full spectra of asymptotic spherical densities of these measures in $\mathbf{P}$, which, in contrast with that of the Sierpinski gasket, consists of a unique interval. These results allow us to compute the irregularity index of $\mathbf{P}$, $\mathcal{I}(\mathbf{P}) \sim 0.6398$, which we define for any self-similar set E with open set condition as $\mathcal{I}(E) = 1 - \frac{C^s(E)}{P^s(E)}$.

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1. Introduction

In general metric spaces, the density of a mass distribution on balls depends on the centre and on the radius of the ball. This is not the case of the Lebesgue measures in $\mathbb{R}^n$, reflecting the homogeneous behaviour of these measures. One of the most basic geometric parameters to consider in the analysis of subspaces of Euclidean spaces is their *dimension*, which turns out to be, in general, a real, non integer, number. A deep result by Marstrand [16] illustrates the extent to which the regularity of the density on balls of the Lebesgue measures is exceptional. If α is a Radon measure on $\mathbb{R}^n$ and

$$\theta^s_\alpha(x, d) = \frac{\alpha(B(x, d))}{(2d)^s} \tag{1.1}$$

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is its s-density on a closed ball, $B(x,r)$, centred at x and with radius d, then the asymptotic density $\lim_{d\to 0}\theta^s_\alpha(x,d)$ can exist and be finite and positive on a set of positive α-measure if and only if s is an integer. In the case of a *self-similar set* E satisfying the *open set condition* (OSC) (see Sec. 3.1) with a non-integer dimension s, like the penta-Sierpinski gasket, it is well known [6] that, if α is a *metric measure* (see Sec. 3.2), the *lower and upper spherical s-densities* of α at $x \in E$, defined by

$$\underline{\theta}^s_\alpha(x) = \liminf_{d\to 0} \theta^s_\alpha(x,d) \text{ and } \overline{\theta}^s_\alpha(x) = \limsup_{d\to 0} \theta^s_\alpha(x,d)$$

respectively, are finite and positive, so they do not coincide in sets of positive α-measure. The local spectrum of s-densities at a point $x \in E$,

$$Spec(\alpha, x) = \left\{ \lim_{d_i \to 0} \theta^s_\alpha(x, d_i) : d_i \text{ is a sequence tending to zero} \right\}, \qquad (1.2)$$

gauges the local regularity of the measure α at x. The spectrum of s-densities of a subset $A \subset E$ is defined as

$$Spec(\alpha, A) = \cup_{x\in A} Spec(\alpha, x). \qquad (1.3)$$

The task of computing such quantities is still out of reach even for the measures in the set $\mathcal{M} := \left\{ \mathcal{H}^s, \mathcal{H}^s_{sph}, C^s, P^s, \mu \right\}$, the Hausdorff, the spherical Hausdorff, the centred Hausdorff, the packing and the invariant measures, respectively (see Secs. 3.2 and 3.3), which are the natural extensions of Lebesgue measures to general metric spaces, and which we call *metric measures*. In order to illustrate the incipient state of knowledge in this respect, we mention that the only connected self-similar set with a non-integer dimension for which the spectra of asymptotic densities of some metric measure are known is the Sierpinski gasket S, for which $\alpha(S)$ and $Spec(\alpha, S)$ for $\alpha \in \{C^s, P^s\}$ were computed in [13], [14] and [19], respectively.

A programme similar to the one developed for S in the literature quoted above turns out to be amenable to any measure $\alpha \in \mathcal{M}$ in the family of *α-exact self-similar sets*, a class of self-similar sets introduced in [19]. A self-similar set E is α-exact if there is an optimal covering or packing element. This means that there is a convex set with maximal density in the case of $\mathcal{H}^s$, and in the cases of the C^s and $\mathcal{H}^s_{sph}$, a ball with maximal density centred in E and centred at $x \in \mathbb{R}^n$, respectively. In the case of P^s, there must exist a ball with minimal density which is centred in E. If a self-similar set with OSC is α-exact, then there also exist optimal coverings (in the case of the measures $\left\{ \mathcal{H}^s, \mathcal{H}^s_{sph}, C^s \right\}$) or packings (in the case of P^s) which can be built from the knowledge of the optimal covering or packing element using the *self-similar tiling principle* (see [18] and [8]).

If a self-similar set E is α-exact, then $\alpha(E)$ can be, in the terminology of [14], *C-computable* for α (i.e., continuous computable), meaning that the task of computation of $\alpha(E)$ can be reduced to the solution of an optimisation problem of a continuous function in a compact domain. Thus, the computation of $\alpha(E)$ is, at least theoretically, amenable. In order to be able to actually estimate $\alpha(E)$, we must develop a computer code whose outputs converge to $\alpha(E)$ giving at each step of the algorithm explicit error intervals. If this task can be accomplished, then we say that E is A-*computable* for α (i.e., algorithmic-computable [14]). See in [10] and [11] that a totally disconnected self-similar set E is an example of a C-computable and A-computable set for $\alpha \in \{C^s, P^s\}$, and see [12] for the issue of the rate of convergence of the estimations of $\alpha(E)$ in that case.

Note that if $\alpha(E)$ is C-computable and A-computable, then, using (3.11), $\alpha(B(x,d))$ and $\theta^s_\alpha(x,d)$ can also be computed.

This research is part of the effort to better understand the basic mathematical models for the phenomena of self-similarity, whose relevance has been widely recognised, since Mandelbrot's seminal book [15], in a growing variety of fields of science and technology, including various branches of Mathematics and Physics, see for example [1], [2] and [3].

2. Results and theoretical implications. An irregularity index

In Sec. 4, we show that the penta-Sierpinski gasket, $\mathbf{P}$, is α-exact for $\alpha \in \{C^s, P^s\}$, and C-computable for $\alpha \in \{\mu, C^s, P^s\}$, where s is the similarity dimension of $\mathbf{P}$. In Sec. 5, we obtain the almost sure pointwise asymptotic spectrum of s-densities of α at the points of $\mathbf{P}$ and the whole spectrum of $\mathbf{P}$ for $\alpha \in \mathcal{M}\lfloor_{\mathbf{P}}$ (see (3.10)). In Sec. 6, we show that $\mathbf{P}$ is A-computable for $\alpha \in \{C^s, P^s\}$. In Sec. 7 we give the estimates for $C^s(\mathbf{P})$ and $P^s(\mathbf{P})$, and for $Spec(\alpha, \mathbf{P})$ for $\alpha \in \{\mu, C^s\lfloor_{\mathbf{P}}, P^s\lfloor_{\mathbf{P}}\}$ obtained using our algorithms.

In particular, in Theorem 5.1 and Corollary 7.1 it is shown that

$$\begin{aligned} Spec(\mu, x) &= \left[\underline{\theta}^s_\mu(x), \overline{\theta}^s_\mu(x)\right] = \left[P^s(\mathbf{P})^{-1}, C^s(\mathbf{P})^{-1}\right] \qquad (2.1) \\ &\sim [0.1476, 0.4098] \end{aligned}$$

holds for μ-*a.e* $x \in \mathbf{P}$. This shows that $\overline{\theta}^s_\mu(x) = C^s(\mathbf{P})^{-1} \sim 0.4098$ and $\underline{\theta}^s_\mu(x) = P^s(\mathbf{P})^{-1} \sim 0.1476$ hold for μ-*a.e* $x \in \mathbf{P}$ (cf. [23]). Therefore, the values of $C^s(\mathbf{P})$ and $P^s(\mathbf{P})$ which, seemingly, only reflect global properties of $\mathbf{P}$, also give an accurate information on the local behaviour of μ-*a.e.* $x \in \mathbf{P}$.

In the case of the value of $C^s(\mathbf{P})$, by Theorem 5 in [9], we also have $\frac{\mu(B(x,r))}{(2r)^s} \leq C^s(\mathbf{P})^{-1}$ for all balls $B(x,r)$ with $x \in \mathbf{P}$ and, by (2.1), the bound is sharp for the family $\{B(x,r) : x \in \mathbf{P}\}$ for μ-*a.e.* $x \in \mathbf{P}$.

Analogously, by (3.5), the value of $P^s(\mathbf{P})^{-1}$ gives a lower bound, $\frac{\mu(B(x,r))}{(2r)^s} \geq P^s(\mathbf{P})^{-1}$, but in this case the inequality is only proved for the typical balls in $\mathcal{B}_\mathcal{O}$ (see (3.4) for a definition), where $\mathcal{O}$ is any feasible open set, i.e., it holds for any ball $B(x,r)$ with $x \in \mathbf{P} - A_1$ and contained in the open decagon $\mathcal{D}$ (see Fig. 1 and the definitions in Sec. 3.1). Again by (2.1), the bound is sharp for the family of typical balls.

That the knowledge of accurate values of $C^s(\mathbf{P})$ and $P^s(\mathbf{P})$ is critically relevant can be seen by the following argument. An s-set $E \subset \mathbb{R}^n$ (i.e. a set with $\mathcal{H}^s(E)$ finite and positive) is called regular if for any Radon measure μ

with $\mu(E) > 0$, $\overline{\theta}^s_\mu(x) = \underline{\theta}^s_\mu(x)$ holds for μ-*a.e.* $x \in E$. A natural way to gauge the irregularity of a self-similar set E with OSC is through the quantity

$$\mathcal{I}(E) := 1 - \tfrac{C^s(E)}{P^s(E)}.$$

If E satisfies the strong open set condition (see Sec. 3.1) and α is a metric measure restricted to E, Theorem 14 in [19] shows that

$$\mathcal{I}(E) = \frac{\overline{\theta}^s_\alpha(x) - \underline{\theta}^s_\alpha(x)}{\overline{\theta}^s_\alpha(x)} \tag{2.2}$$

for α-a.e. $x \in E$. Note that this index can be applied in a more general setting, namely, if α is a Radon measure on a set E for which $\underline{\theta}^s_\alpha(x)$ and $\overline{\theta}^s_\alpha(x)$ are constant for α-a.e. $x \in E$, then (2.2) holds by Theorem 3.1 in [25].

There are some disconnected self-similar sets E for which the values of $C^s(E)$ and $P^s(E)$ are known and therefore $\mathcal{I}(E)$ can be obtained. Let $C_1(r)$, $C_2(r)$ and $S(r)$ be the r-depending Cantor sets in $\mathbb{R}$, in $\mathbb{R}^2$, and the Sierpinski sets in $\mathbb{R}^2$, respectively, where r is the contraction ratio of the homotheties that generate such sets. Then, $\mathcal{I}(C_1(r)) = 0.5$ for $0 < r \leq \frac{1}{3}$, $\mathcal{I}(C_2(\frac{1}{4})) \sim 0.6743$ and $\mathcal{I}(S(\frac{1}{3})) \sim 0.6141$ (see [12] and the references [3], [6], [25] and [26] in that paper for these and other examples).

In the case of $\mathbf{P}$, using the results in Sec. 7, we have $\mathcal{I}(\mathbf{P}) \sim 0.6398$ which means that $\mathcal{I}(\mathbf{P}) > \mathcal{I}(S) \sim 0.3977$, where S stands for the classical Sierpinski gasket, i.e., $r = 0.5$ (see the estimates for $C^s(S)$ and $P^S(S)$ in [19]).

The reader can compare $\mathcal{I}(E)$ with the coefficient of irregularity of a fractal set E defined by Tricot in [25] as the difference between the packing and the Hausdorff dimension of E, that in the case of self-similar sets with OSC is zero because these dimensions coincide.

3. Preliminaries

3.1. The penta-Sierpinski gasket P

Consider the circle with centre $\mathbf{O} := (0,0)$ and radius $\beta := (2\sin(\frac{\pi}{5}))^{-1}$ and let $z := (0, \beta)$. Let P be a regular pentagon with unit length inscribed in such a circle and with vertexes $z_i = g_{(i+1)\alpha}(z)$, $i \in M := \{0, ..., 4\}$, $\alpha := \frac{2\pi}{5}$, and where g_α is the counterclockwise rotation with centre at $\mathbf{O}$ and angle α (see Fig. 1). The penta-Sierpinski gasket $\mathbf{P}$ is the self-similar set generated by the system of five homotheties $\Psi = \{f_i\}_{i\in M}$, where $f_i(x) = rx + (1-r)z_i$, $i \in M$, $r = \frac{\phi}{2\phi+1}$ is the contraction ratio of f_i and $\phi := \frac{1+\sqrt{5}}{2}$ is the golden number. The diameter of P is $diam(P) = \phi = diam(\mathbf{P})$. The copies $f_i\mathbf{P}$, $i \in M$ are just-touching, i.e., $f_i\mathbf{P} \cap f_{(i+1)mod(5)}\mathbf{P}$, $i \in M$ are singletons (see Fig. 1).

Let F be *the Hutchinson operator* defined by $FA := \cup_{i\in M} f_i A$, $A \subset \mathbb{R}^2$. It is well known, see [6], that $\mathbf{P}$ is the unique non-empty compact set, *invariant* for F, i.e., that admits the self-similar decomposition

$$\mathbf{P} = \cup_{i\in M} f_i\mathbf{P} = F\mathbf{P}.$$

The set $\mathbf{P}$ is called the *attractor of* Ψ under the contracting operator F, because $\lim_{k\to\infty} F^k(A) = \mathbf{P}$ holds for any non-empty compact subset $A \subset \mathbb{R}^2$, where $F^k = F \circ F \circ ... \circ F$ is the k-th iterate of F and the convergence is with respect to the Hausdorff metric (see [6]). Furthermore, if $A \subset \mathbf{P}$, then $F^k(A) \subset \mathbf{P}$ for any $k \in \mathbb{N}^+$. We use these facts in our algorithms, where we take as the initial compact set $A_1 := \{z_i\}_{i\in M}$ (note that $z_i \in M$ are the fixed points of the similitudes in Ψ), and we obtain the set

$$A_k := F^{k-1}(A_1) \subset \mathbf{P},\ k \geq 2 \tag{3.1}$$

as a discrete approximation of $\mathbf{P}$ at the iteration k.

The system of similitudes Ψ satisfies the OSC (see [21]), meaning there is an open set $\mathcal{O} \subset \mathbb{R}^2$ such that $f_i(\mathcal{O}) \subset \mathcal{O}$ for all $i \in M$ and $f_i(\mathcal{O}) \cap f_j(\mathcal{O}) = \varnothing$ for $i,\, j \in M,\ i \neq j$. We will refer to such a set $\mathcal{O}$ as a *feasible open set* (for $\mathbf{P}$). Furthermore, if $\mathcal{O} \cap \mathbf{P} \neq \varnothing$, then it is said that $\mathcal{O}$ satisfies the strong open set condition SOSC (cf. [7], [20] and [24]). A feasible open set $\mathcal{D}$ that fulfils the SOSC is the interior of the regular decagon that shares five vertexes with $\mathbf{P}$ (see Fig. 1). Notice that any enlargement of $\mathcal{D}$ would cause its copies under the similitudes in Ψ to overlap among themselves. The similarity dimension of $\mathbf{P}$, i.e., the unique solution to the equation $\sum_{i\in M} r_i^x = 5r^x = 1$ is $s = -\frac{\log 5}{\log r}$ and, as Ψ satisfies the OSC, s is also the α-dimension of $\mathbf{P}$, $\alpha \in \left\{\mathcal{H}^s, \mathcal{H}^s_{sph}, C^s, P^s\right\}$ (see [17] and [5] for these and other definitions of dimension of a set).

To denote the compositions $f_i := f_{i_1} \circ f_{i_2} \circ ... \circ f_{i_k}$, we use the multiindexes $i := i_1, i_2, ..., i_k \in M^k$, and we write r_i for the contraction ratio of f_i (which equals to r^k if $i \in M^k$). The set $\mathbf{P}$ can be parameterised as $\mathbf{P} = \{\pi(i) : i \in \Sigma\}$ with parameter space $\Sigma := M^\infty$ and *geometric projection mapping* $\pi : \Sigma \to \mathbf{P}$ given by $\pi(i) = \cap_{k=1}^\infty f_{i(k)}(\mathbf{P})$, where $i(k) \in M^k$ denotes the k-th curtailment $i_1 i_2 \dots i_k$ of $i = i_1 i_2 \cdots \in \Sigma$. Notice that π is non-injective. We adopt the convention $M^0 = \emptyset$ and write $M^* = \cup_{k=0}^\infty M^k$ for the set of words of finite length. For any $i \in M^*$, the cylinder sets are denoted by $\mathbf{P}_i := f_i(\mathbf{P})$, and $f_i(\mathbf{P}) := \mathbf{P}$ if $i \in M^0$. Sometimes we also write A_i for $f_i(A)$, $A \subset \mathbb{R}^2$, $i \in M^*$. For $i \in M^k$, $\mathbf{P}_i$ is a *cylinder of the* k-th generation, or k-cylinder (see $\mathbf{P}_i$, $i \in M$ in Fig. 1). We may identify each $i \in M^k$ with the k-cylinder set in Σ, $\{j \in \Sigma : j(k) = i\}$. Then, $\pi(i) = \mathbf{P}_i$.

The shift map σ is defined by $\sigma(i) := i_2 i_3 ....$ for $i \in \Sigma$, and the geometric shift correspondence $\mathcal{T} : \mathbf{P} \to \mathbf{P}$ as

$$\mathcal{T}(z) = \pi \circ \sigma \circ \pi^{-1}(z). \tag{3.2}$$

The shift orbit of $x \in \mathbf{P}$ is given by $\left\{\mathcal{T}^k(x) : k \in \mathbb{N}\right\}$.

3.2. Metric measures and typical balls

Let $A \in \mathbb{R}^n$, the *Hausdorff centred measure*, $C^s(A)$, the *spherical centred Hausdorff measure*, $C^s_{sph}(A)$, and the *Hausdorff measure*, $\mathcal{H}^s(A)$, of A are defined (see [22]) and [4]) through optimal coverings of A, and the *packing measure*, $P^s(A)$ (see [22]) through optimal packings of A.

The reader can check in [19] the relevant role played in the analysis of the local structure of self-similar sets E with OSC by a distinguished family $\mathcal{B}$ of balls, the *typical balls*:

$$\mathcal{B} := \{B(x,d) : \ x \in E, \ B(x,d) \subset \mathcal{O} \text{ for some feasible open set } \mathcal{O} \text{ for } E\} . \tag{3.3}$$

Given a feasible open set $\mathcal{O}$ for E, we define

$$\mathcal{B}_{\mathcal{O}} : = \{B(x,d) \in \mathcal{B} : B(x,d) \subset \mathcal{O}\} . \tag{3.4}$$

The metric measures mentioned above have a much simpler expression when dealing with a self-similar set E with OSC for an open set $\mathcal{O}$ and with similarity dimension s. In the case of $C^s(E)$ and $P^s(E)$, the browsing for optimal coverings or packings can be reduced to the search of balls of optimal density within the class of typical balls. In particular, it is known, [18], that

$$P^s(E) = \left(\inf\left\{\theta^s_\mu(x,d) : B(x,d) \in \mathcal{B}_{\mathcal{O}}\right\}\right)^{-1}, \tag{3.5}$$

and in [19], Lemma 13, it is proved that

$$C^s(E) = \left(\sup\left\{\theta^s_\mu(x,d) : B(x,d) \in \mathcal{B}_{\mathcal{O}}\right\}\right)^{-1}, \tag{3.6}$$

where μ is the invariant measure for the Markov operator (see Sec. 3.3).

3.3. The Markov operator and the invariant measure on P

It is known that the Markov operator $\mathbf{M} : P(\mathbb{R}^2) \longrightarrow P(\mathbb{R}^2)$ defined in the space $P(\mathbb{R}^2)$ of compactly supported probability Borel measures of $\mathbb{R}^2$ as $\mathbf{M}(\alpha) = r^s \sum_{i=0}^{4} \alpha \circ f_i^{-1}$ is contractive with respect to a suitable metric (see [6] and [1]), where $\{f_i\}_{i\in M}$ are the homotheties that generate $\mathbf{P}$. Its unique fixed point, μ, is supported on $\mathbf{P}$, and for any $\alpha \in P(\mathbb{R}^2)$,

$$\mathbf{M}^k(\alpha) = r^{ks} \sum_{i\in\mathbf{M}^k} \alpha \circ f_i^{-1} \xrightarrow{w} \mu \tag{3.7}$$

holds, where $\xrightarrow{w}$ denotes the weak convergence of measures and $\mathbf{M}^k = \mathbf{M} \circ \mathbf{M} \circ ... \circ \mathbf{M}$ is the k-th iterate of $\mathbf{M}$. The measure μ is called the invariant or natural probability measure associated with the Markov operator $\mathbf{M}$. If we take $\mu_1 := \frac{1}{5}\sum_{i=0}^{4} \delta_{z_i}$ as the initial measure in (3.7), and using that $r^{ks} = 5^{-k}$, we get that

$$\mu_k := \mathbf{M}^{k-1}(\mu_1) = \frac{1}{5^{k-1}} \sum_{i\in\mathbf{M}^{k-1}} \mu_1 \circ f_i^{-1} = \frac{1}{5^k} \sum_{i\in\mathbf{M}^{k-1}} \sum_{j=0}^{4} \delta_{f_i(z_j)} \xrightarrow[k\to\infty]{w} \mu. \tag{3.8}$$

Notice that, for any $k \geq 1$, the discrete probability measure μ_k is supported by the set A_k (see (3.1)).

It is known (see [6]) that μ coincides with the projection of ν on $\mathbf{P}$,

$$\mu = \nu \circ \pi^{-1}, \tag{3.9}$$

where ν is the Bernoulli measure on Σ that gives weight $\frac{1}{5}$ to each symbol in M. By (3.9), we know that $\mu(\mathbf{P}_i) = r^k$ for $i \in M^k$. On the other hand, any metric measure $\alpha \in \mathcal{M}$ scales under similitudes, so $\alpha(\mathbf{P}_i) = r^k \alpha(\mathbf{P})$, for $i \in M^k$. Since μ and α are multiples on cylinder sets, they are indeed multiple measures. Then, all the normalised measures $(\alpha(\mathbf{P}))^{-1} \alpha\lfloor_{\mathbf{P}}$ coincide with μ and with the normalised Hausdorff measure $\mu^* := \frac{1}{\mathcal{H}^s(\mathbf{P})}\mathcal{H}^s\lfloor_{\mathbf{P}}$. Here $\beta\lfloor_{\mathbf{P}}$ stands for a measure β restricted to $\mathbf{P}$.

Remark 3.1. From now on we shall work with μ rather than with other metric measure on $\mathbf{P}$, keeping in mind that, for all $\alpha \in \mathcal{M}\lfloor_{\mathbf{P}}$ where

$$\mathcal{M}\lfloor_{\mathbf{P}} := \left\{\mathcal{H}^s\lfloor_{\mathbf{P}}, \mathcal{H}^s_{sph}\lfloor_{\mathbf{P}}, C^s\lfloor_{\mathbf{P}}, P^s\lfloor_{\mathbf{P}}, \mu\right\} \tag{3.10}$$

and for any Borel set A, we have

$$\alpha(A) = \alpha(\mathbf{P})\mu(A), \tag{3.11}$$

so the computation of $\alpha(A)$ boils down to the computation of $\alpha(\mathbf{P})$ plus the computation of $\mu(A)$.

4. C-computability of $P^s(\mathbf{P})$ and $C^s(\mathbf{P})$

Our algorithms for the estimation of $P^s(\mathbf{P})$ and $C^s(\mathbf{P})$ are based on the following theorem. The boundary of a set A is denoted by ∂A. See in Sec. 3.1 the definition of the open set $\mathcal{D}$. The distances ρ and δ can be seen in Fig. 1.

Theorem 4.1. *Let $\mathbf{P}$ be the penta-Sierpinski gasket, $\rho = dist(\mathbf{P}_{01}, \partial\mathcal{D})$ and $\delta = dist(\mathbf{P}_{01}, \mathbf{P} - \mathbf{P}_0)$. Then,*
(i) $P^s(\mathbf{P}) = \max\left\{\frac{(2d)^s}{\mu(B(x,d))} : x \in \mathbf{P}_{01} \text{ and } r\rho \le d \le \rho\right\}$
(ii) $C^s(\mathbf{P}) = \min\left\{\frac{(2d)^s}{\mu(B(x,d))} : x \in \mathbf{P}_{02} \text{ and } r\delta \le d \le \delta\right\}$

In order to prove this theorem, we need Lemmas 4.3 and 4.4 below and the following definition.

Definition 4.2. We say that the balls $B(x,d)$ and $B(y,r)$ are density-equivalent if $\theta^s_\mu(x,d) = \theta^s_\mu(y,r)$.

Lemma 4.3. *Let E be a self-similar set generated by the system of similarities of $\mathbb{R}^n$, $\Psi = \{f_i\}_{i\in M}$, with contracting ratios $\{r_i\}_{i\in M}$, $M := \{0, 1, ..., m-1\}$, and let $h : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a similarity with scaling factor r_h. Then,*
(i) If $A \subset \mathbb{R}^n$ is such that $h(A \cap E) = h(A) \cap E$, then $\mu(h(A)) = r_h^s \mu(A)$.
(ii) If $C \subset E$ is such that $h(C) \subset E$, then

$$\mu(h(B(x,d) \cap C)) = r_h^s \mu(B(x,d) \cap C).$$

Let $\mathcal{O}$ be a feasible open set for E, and let $\mathcal{B}_\mathcal{O}$ be the set of balls centred at points of E and contained in $\mathcal{O}$.
(iii) If $A \subset \mathcal{O}$ is μ-measurable, then $\mu(f_i A) = r_i^s \mu(A)$, $i \in M^k$.
(iv) If $B(x,d) \in \mathcal{B}_\mathcal{O}$, then the balls $B(x,d)$ and $B(f_i x, r_i d)$, $i \in M^k$ are density-equivalent.
(v) If $i \in M^k$ and $C \subset \mathcal{O}_i$ is μ-measurable, then $\mu(f_i^{-1}C) = r_i^{-s}\mu(C)$.

(vi) If $i \in M^k$ and $B(x,d) \in f_i(\mathcal{B}_\mathcal{O})$, then $B(f_i^{-1}x, r_i^{-1}d) \in \mathcal{B}_\mathcal{O}$ and the balls $B(x,d)$ and $B(f_i^{-1}x, r_i^{-1}d)$ are density-equivalent.
(vii) $\theta_\mu^s(f_i x, r_i d) \geq \theta_\mu^s(x,d)$, $i \in M^k$, $k \geq 1$.
(viii) $\theta_\mu^s(f_i^{-1}x, r_i^{-1}d) \leq \theta_\mu^s(x,d)$, $i \in M^k$, $k \geq 1$.

Proof. Parts (i) and (ii) are proved in Lemma 4 of [14], and Parts (iii), (iv), (v) and (vi) are proved in Lemma 10 of [19].
Part (vii) is proved using (v) with $C = B(f_i x, r_i d) \cap \mathcal{O}_i$. We get that

$$\begin{aligned}\mu(B(f_i x, r_i d)) &= \mu(B(f_i x, r_i d) \cap \mathcal{O}_i) + \mu(B(f_i x, r_i d) \cap \mathcal{O}_i^c) \\ &\geq \mu(B(f_i x, r_i d) \cap \mathcal{O}_i) = r_i^s \mu(B(x,d) \cap \mathcal{O}) = r_i^s \mu(B(x,d))\end{aligned}$$

holds, so $\theta_\mu^s(f_i x, r_i d) \geq \theta_\mu^s(x,d)$.
Part (viii) follows replacing $B(x,r)$ in (vii) with $B(f_i^{-1}x, r_i^{-1}d)$. $\square$

Lemma 4.4. *Let $B(x,d)$ with $x \in \mathbf{P}$, and let $\mathcal{D}$ be the feasible open set defined in Sec.* 3.1 *(see Fig. 1).*
(i) There is a ball $B(y,d)$ with $y \in \mathbf{P}_0$ which is density-equivalent to $B(x,d)$. If $B(x,d) \subset \mathcal{D}$, then $B(y,d) \subset \mathcal{D}$.
(ii) If $B(x,d) \subset \mathcal{D}$ and $d \leq \rho = dist(\mathbf{P}_{01}, \partial\mathcal{D})$, then there is a ball $B(z,d) \subset \mathcal{D}$ with $z \in \mathbf{P}_{01}$, and $\theta_\mu^s(z,d) \leq \theta_\mu^s(x,d)$.
(iii) If $d \leq \delta = dist(\mathbf{P}_{01}, \mathbf{P}_1)$, then there is a ball $B(z,d)$ with $z \in \mathbf{P}_{02}$, and $\theta_\mu^s(z,d) \geq \theta_\mu^s(x,d)$.

Proof. (i) Let g_α be the counterclockwise rotation of centre $\mathbf{O} = (0,0)$ and angle $\alpha = \frac{2\pi}{5}$. Let $i \in M$ and $x \in \mathbf{P}_i$. For $k \in \mathbb{Z}$, we know that $g_{k\alpha}\mathbf{P} = \mathbf{P}$ and $g_{k\alpha}(B(x,d) \cap \mathbf{P}) = B(g_{k\alpha}(x), d) \cap \mathbf{P}$ holds. Then, Lemma 4.3 (i) guarantees that $B(x,d)$ and $B(y,d)$, $y \in \mathbf{P}_0$ are density-equivalent, where $y = g_{(5-i)\alpha}(x)$. Since, for $k \in \mathbb{Z}$, $g_{k\alpha}\mathcal{D} = \mathcal{D}$ holds, if $B(x,d) \subset \mathcal{D}$, then $B(y,d) \subset \mathcal{D}$ also holds.
(ii) Let $B(x,d) \subset \mathcal{D}$ with $x \in \mathbf{P}$ and $d \leq \rho$. Using Part (i), we know that there is a ball $B(y,d) \subset \mathcal{D}$ with $y \in \mathbf{P}_0$ which is density-equivalent to $B(x,d)$. We shall prove that if $y \in \mathbf{P}_{00} \cup \mathbf{P}_{04}$, there is a ball $B(z,d) \subset \mathcal{D}$ with $z \in \mathbf{P}_{01}$ which is density-equivalent to $B(y,d)$, and that if $y \in \mathbf{P}_{02} \cup \mathbf{P}_{03}$, then there is a ball $B(z,d) \subset \mathcal{D}$ with $z \in \mathbf{P}_{01}$ and $\theta_\mu^s(z,d) \leq \theta_\mu^s(y,d)$.

The counterclockwise rotations $g_{k\alpha}^0$ with centre $\mathbf{O}_0 = f_0\mathbf{O}$ and angle $k\alpha$ satisfy $g_{k\alpha}^0\mathbf{P}_0 = \mathbf{P}_0$ for all $k \in \mathbb{Z}$. Furthermore, $g_\alpha^0\mathbf{P}_{00} = \mathbf{P}_{01}$ and $g_{2\alpha}^0\mathbf{P}_{04} = \mathbf{P}_{01}$ hold. Now, if $y \in \mathbf{P}_{00}$, then $B(y,d) \cap \mathbf{P} = B(y,d) \cap \mathbf{P}_0$ and $g_\alpha^0 B(y,d) = B(z,d) \subset \mathcal{D}$, where $z := g_\alpha^0(y) \in \mathbf{P}_{01}$. As $d \leq \rho < \delta := dist(\mathbf{P}_{01}, \mathbf{P} - \mathbf{P}_0)$, it follows that $B(z,d) \cap \mathbf{P} = B(z,d) \cap \mathbf{P}_0$. Therefore,

$$g_\alpha^0(B(y,d) \cap \mathbf{P}) = g_\alpha^0(B(y,d) \cap \mathbf{P}_0) = B(z,d) \cap \mathbf{P}_0 = B(z,d) \cap \mathbf{P},$$

and Lemma 4.3 (i) guarantees that $B(y,d)$ and $B(z,d)$ are density-equivalent. If $y \in \mathbf{P}_{04}$ and $d \leq \rho$, the same argument is valid if we replace g_α^0 with $g_{2\alpha}^0$. Now, let $y \in \mathbf{P}_{03}$ and consider the ball $B(z,d) := g_{3\alpha}^0 B(y,d)$, with $z := g_{3\alpha}^0(y) \in \mathbf{P}_{01}$. Notice that $B(z,d) \cap \mathbf{P} = B(z,d) \cap \mathbf{P}_0$ holds since $d \leq \delta$. Using Lemma 4.3 (ii) with $h = g_{3\alpha}^0$, $C = \mathbf{P}_0$ and the ball $B(y,d)$, we obtain that

$$\begin{aligned}\mu(B(z,d)) &= \mu(B(z,d) \cap \mathbf{P}) = \mu(B(z,d) \cap \mathbf{P}_0) \\ &= \mu(g_{3\alpha}^0(B(y,d) \cap \mathbf{P}_0)) = \mu(B(y,d) \cap \mathbf{P}_0).\end{aligned}$$

Then,

$$\begin{aligned}\mu(B(y,d)) &= \mu\left(B(y,d)\cap \mathbf{P}_0\right)+\mu\left(B(y,d)\cap(\mathbf{P}-\mathbf{P}_0)\right)\\ &\geq \mu\left(B(y,d)\cap \mathbf{P}_0\right)=\mu(B(z,d)),\end{aligned}$$

so $\theta^s_\mu(z,d)\leq \theta^s_\mu(y,d)$. Lastly, if $y\in \mathbf{P}_{02}$, we consider the symmetry t with respect to the axis of the pentagon P through $\mathbf{O}$ and $\mathbf{O}_0$. We have that $t\mathbf{P}=\mathbf{P}$ and $t(B(y,d)\cap\mathbf{P})=B(ty,d)\cap\mathbf{P}$. Using Lemma 4.3 (i), we get that $B(ty,d)$ is density-equivalent to $B(y,d)$, and $ty\in\mathbf{P}_{03}$. Since $d\leq\rho$, we can now apply the argument above for balls centred in $\mathbf{P}_{03}$ and find a ball $B(z,d)$ with $z\in\mathbf{P}_{01}$ and with $\theta^s_\mu(z,d)\leq\theta^s_\mu(y,d)$.
(iii) The role of the upper bound ρ in the proof of Part (ii) is twofold. First, it guarantees that if the radius d of a ball $B(x,d)$ satisfies $d\leq\rho$, then $B(x,d)\subset\mathcal{D}$. Second, it guarantees that if $x\in\mathbf{P}_{01}$, then $B(x,d)\cap(\mathbf{P}-\mathbf{P}_0)=\varnothing$, which also holds if $d\leq\delta$. If we drop the assumption $d\leq\rho$, we still get that $\theta^s_\mu(z,d)=\theta^s_\mu(y,d)$ for any ball $B(y,d)$ with $y\in\mathbf{P}_{00}\cup\mathbf{P}_{04}$ and some ball $B(z,d)$ with $z\in\mathbf{P}_{01}$. In this case, $B(z,d)\subset\mathcal{D}$ is not guaranteed, but this is not required for the statement in (iii). By the argument given in the proof of (ii), $\theta^s_\mu(z,d)\leq\theta^s_\mu(y,d)$ if $z\in\mathbf{P}_{01}$ and $d\leq\delta$ and $y=g^0_{-3\alpha}(z)\in\mathbf{P}_{03}$. Therefore, if $d\leq\delta$, $\theta^s_\mu(z,d)\leq\theta^s_\mu(y,d)$ holds for any ball $B(z,d)\in\mathbf{P}_{00}\cup\mathbf{P}_{04}\cup\mathbf{P}_{01}$ and some ball $B(y,d)$ with $y\in\mathbf{P}_{03}$. Lastly, by the argument given in the proof of (ii), $\theta^s_\mu(z,d)=\theta^s_\mu(y,d)$ holds if $z\in\mathbf{P}_{02}$ and $y=tz\in\mathbf{P}_{03}$, and the proof of (iii) is complete. □

Proof of Theorem 4.1. (i) Since $\mathcal{D}$ satisfies the open set condition for the system of similitudes Ψ, we know by (3.5) and Lemma 4.4 (i) and (ii), that

$$P^s(\mathbf{P})=\sup\left\{\tfrac{(2d)^s}{\mu(B(x,d))}:\ x\in\mathbf{P}_{01},\ B(x,d)\in\mathcal{B}_\mathcal{D}\right\}.$$

Let $B(x_0,d)\in\mathcal{B}_\mathcal{D}$ with $x_0\in\mathbf{P}_{01}$ and assume that $d>\rho$. Let k be such that $d_k:=r^kd\leq\rho<r^{k-1}d$. Lemma 4.3 (iv) guarantees that the ball $f_0^k(B(x_0,d))=B(f_0^k(x_0),d_k)\subset\mathcal{D}$ is density-equivalent to $B(x_0,d)$. Using Lemma 4.4 (ii), we get that there is a ball $B(z,d_k)$ with $z\in\mathbf{P}_{01}$, and $\theta^s_\mu(z,d_k)\leq\theta^s_\mu(x_0,d)$. Then,

$$P^s(\mathbf{P})=\sup\left\{\tfrac{(2d)^s}{\mu(B(x,d))}:\ x\in\mathbf{P}_{01},\ d\leq\rho\right\}.$$

Let $x_0\in\mathbf{P}_{01}$ and $d\leq\rho$. If $d<r\rho$, then $B(x_0,d)\in f_0\mathcal{B}_\mathcal{D}$, and using Lemma 4.3 (vi), we get that the ball $B(y_1,d_1)\in\mathcal{B}_\mathcal{D}$ with $y_1:=f_0^{-1}x_0\in\mathbf{P}_1$, and $d_1:=r^{-1}d<\rho$ is density-equivalent to $B(x_0,d)$. Lemma 4.4 (ii) guarantees that there is a ball $B(x_1,d_1)\in\mathcal{B}_\mathcal{D}$ with $x_1\in\mathbf{P}_{01}$ and $\theta^s_\mu(x_1,d)\leq\theta^s_\mu(y_1,d)$. If $d_1<r\rho$, we can repeat the argument above k times, obtaining a ball $B(x_k,d_k)\in\mathcal{B}_\mathcal{D}$ with a density lower than or equal to $B(x_0,d)$, with $x_k\in\mathbf{P}_{01}$ and $d_{k-1}<r\rho\leq r^{-1}d_{k-1}=:d_k$. As $r\rho\leq d_k<\rho$ and $\frac{(2d)^s}{\mu(B(x,d))}$ is a continuous function on the compact set $\mathbf{P}_{01}\times[r\rho,\rho]$, the proof of (i) is completed.
(ii) By (3.6)

$$C^s(\mathbf{P}) = \inf\left\{\tfrac{(2d)^s}{\mu(B(x,d))} :\ B(x,d) \in \mathcal{B}_{\mathcal{D}}\right\}$$

and, by Lemma 4.3 (vii), we have

$$C^s(\mathbf{P}) = \inf\left\{\tfrac{(2d)^s}{\mu(B(x,d))} : x \in \mathbf{P},\ 0 < d \leq \delta\right\}.$$

Using Parts (i) and (iii) of Lemma 4.4, we get

$$C^s(\mathbf{P}) = \inf\left\{\tfrac{(2d)^s}{\mu(B(x,d))} :\ x \in \mathbf{P}_{02},\ 0 < d \leq \delta\ \right\}.$$

Let $B(x,d)$ with $x \in \mathbf{P}_{02}$, and $0 < d \leq \delta$. Assume first that $d < r\delta$ and $x \in \mathbf{P}_{02} - \mathbf{P}_{022}$. Since $dist(\mathbf{P}_{02} - \mathbf{P}_{022}, \mathbf{P}_1) = dist(\mathbf{P}_{021}, \mathbf{P}_{140}) = r\delta$, using that $f_0^{-1}\mathbf{P}_0 \subset \mathbf{P}$ and Lemma 4.3 (ii), we see that

$$\begin{aligned} \mu(B(x,d)) &= \mu(B(x,d) \cap \mathbf{P}_0) = r^{-s}\mu(B(f_0^{-1}x, r^{-1}d) \cap \mathbf{P}) \\ &= r^{-s}\mu(B(f_0^{-1}x, r^{-1}d)), \end{aligned}$$

so $B(f_0^{-1}x, r^{-1}d)$ and $B(x,d)$ are density-equivalent. Using Lemma 4.4 (i) and (iii), we get a ball $B(y_1, r^{-1}d)$ with $y_1 \in \mathbf{P}_{02}$ and with a density greater than or equal to that of the balls $B(f_0^{-1}x, r^{-1}d)$ and $B(x,d)$.

Assume now that $d \leq r\delta$ and $x \in \mathbf{P}_{022}$. The homothety h^{-1} with fixed point at $\mathbf{P}_0 \cap \mathbf{P}_1$ and contraction constant r^{-1} bijectively maps $\mathbf{P}_{02} \cup \mathbf{P}_{14}$ onto $\mathbf{P}_0 \cup \mathbf{P}_1$. Using Lemma 4.3 (ii) with $C = \mathbf{P}_{02} \cup \mathbf{P}_{14}$, that $B(x,d) \cap (\mathbf{P}_{02} \cup \mathbf{P}_{14}) = B(x,d) \cap \mathbf{P}$, and that

$$\begin{aligned} B(h^{-1}x, r^{-1}d) \cap h^{-1}(\mathbf{P}_{02} \cup \mathbf{P}_{14}) &= B(h^{-1}x, r^{-1}d) \cap (\mathbf{P}_0 \cup \mathbf{P}_1) \\ &= B(h^{-1}x, r^{-1}d) \cap \mathbf{P}, \end{aligned}$$

we get that $B(h^{-1}x, r^{-1}d)$ and $B(x,d)$ are density-equivalent. Observe also that, since $h\mathbf{P}_{02} = \mathbf{P}_{022}$, we see that $h^{-1}x \in \mathbf{P}_{02}$.

We have proved that if $x \in \mathbf{P}_{02}$ and $d < r\delta$, we can find a new ball centred in $\mathbf{P}_{02}$, with radius $r^{-1}d$, and with a density greater than or equal to that of the ball $B(x,d)$. If $r^{-1}d < r\delta$, we can repeat the same argument k times until $\delta \geq r^{-k}d \geq r\delta$. Then,

$$C^s(\mathbf{P}) = \inf\left\{\tfrac{(2d)^s}{\mu(B(x,d))} : x \in \mathbf{P}_{02},\ r\delta \leq d \leq \delta\right\}.$$

As $\frac{(2d)^s}{\mu(B(x,d))}$ is a continuous function on the compact set $\mathbf{P}_{02} \times [r\delta, \delta]$, the proof is complete. □

Corollary 4.5. *The set* $\mathbf{P}$ *is* C^s *and* P^s*-exact and the packing and centred Hausdorff measures of* $\mathbf{P}$ *are* C*-computable.*

Proof. The exactness of $\mathbf{P}$ w.r.t. both measures means that there are optimal balls. This is a consequence of Theorem 4.1, which also imply that the computation of both $C^s(\mathbf{P})$ and $P^s(\mathbf{P})$ is the solution of an optimisation problem of a continuous function on a compact domain, so they are C-computable. □

5. Asymptotic density spectra of P

Let $\alpha \in \mathcal{M}\lfloor_{\mathbf{P}}$ (see 3.10) (recall that, from (3.11), μ coincides with the normalisation of any measure in $\mathcal{M}\lfloor_{\mathbf{P}}$ to a probability measure) and let

$$Im(\theta^s_\alpha, \mathcal{B}) = \{\theta^s_\alpha(x,d) : B(x,d) \in \mathcal{B}\},$$

where $\mathcal{B}$ is the set of typical balls (see (3.3)).

In [19], Theorem 14 and Corollary 15, we obtain the α-almost sure pointwise spectra $Spec(\alpha, x)$ for all $\alpha \in \mathcal{M}\lfloor_{\mathbf{E}}$, for general self-similar sets $\mathbf{E}$ in $\mathbb{R}^n$ with OSC, together with several related results that we now compile, applied to the penta-Sierpinski gasket, in the following theorem.

Recall the definitions of $Spec(\alpha, x)$, $Spec(\alpha, A)$ of a subset $A \in \mathbf{P}$, and of the geometric shift correspondence $\mathcal{T}$ (see (1.2), (1.3), and (3.2), respectively).

Theorem 5.1. *Let $\widehat{\mathbf{P}}$ be the full measure set*

$$\widehat{\mathbf{P}} := \left\{y \in \mathbf{P} : \left\{T^k(y) : k \in \mathbb{N}\right\} \text{ is dense in } \mathbf{P}\right\},$$

let $\mathcal{O}$ be a feasible open set for $\mathbf{P}$ and let $\alpha \in \mathcal{M}^s\lfloor_{\mathbf{P}}$. Then,
(i) For any $y \in \widehat{\mathbf{P}}$,

$$Spec(\alpha, y) = Spec(\alpha, \widehat{\mathbf{P}}) = Spec(\alpha, \mathcal{O} \cap \mathbf{P}) = Im(\theta^s_\alpha, \mathcal{B}) = \left[\frac{\alpha(\mathbf{P})}{P^s(\mathbf{P})}, \frac{\alpha(\mathbf{P})}{C^s(\mathbf{P})}\right]. \tag{5.1}$$

(ii)

$$Spec(\alpha, \mathbf{P}) = \left[\frac{\alpha(\mathbf{P})}{P^s(\mathbf{P})}, \frac{\alpha(\mathbf{P})}{C^s(\mathbf{P})}\right] \tag{5.2}$$

Proof. The assertions in Part (i) are a particularisation of Theorem 14 and Corollary 15 in [19] to $\mathbf{P}$. We now prove Part (ii). Take the regular open decagon $\mathcal{D}$ of Fig. 1 as $\mathcal{O}$ in (5.1). We may write

$$\mathbf{P} = (\mathcal{D} \cap \mathbf{P}) \cup (\mathbf{P} - \mathcal{D}) = \mathcal{D} \cup A_1$$

(see (3.1)). Then, $Spec(\alpha, \mathbf{P}) = Spec(\alpha, \mathcal{D} \cap \mathbf{P}) \cup Spec(\alpha, A_1)$. Notice now that the balls centred at the points in A_1 and with a radius small enough are density-equivalent, by a suitable rotation, to balls centred at points in $\mathcal{D} \cap \mathbf{P}$, so, from Part (i), $Spec(\alpha, A_1) \subset Spec(\alpha, \mathcal{D} \cap \mathbf{P})$, and Part (ii) follows. □

Remark 5.2. Notice that, in contrast to the spectra of the Sierpinski gasket (see Corollary 32 in [19]), that consist of two disjoint intervals, the α-spectra of $\mathbf{P}$, $\alpha \in \mathcal{M}^s\lfloor_{\mathbf{P}}$, consist of a unique interval.

6. A-computability of $P^s(\mathbf{P})$ and $C^s(\mathbf{P})$

The algorithms we use for the results in Sec. 7 work with the sets $A_k \subset \mathbf{P}$ and with the measures μ_k as approximations of $\mathbf{P}$ and μ at iteration k, respectively (see (3.1) and (3.8)). In such algorithms, we generate for $k = 1, 2, 3, ...,$ the set of points A_k and compute the μ_k-densities of all balls $B(x,d)$ centred at points $x \in A_k$, with radius in the relevant range and with $d = |x-y|$, $y \in A_k$. Our proxy for the optimal densities are the balls with optimal μ_k-densities among these discretised collections of balls. In this section, we show that this procedure allows us the computation of the optimal μ-densities with error intervals of arbitrary small length, which show that $P^s(\mathbf{P})$ and $C^s(\mathbf{P})$ are A-computable. The following lemma is the basic tool for the comparison of the measures μ and μ_k.

Lemma 6.1.
(i) Let $\left\{\mathbf{P}_i : i \in I \subset M^k\right\}$, $k \in \mathbb{N}^+$, *be a collection of* k*-cylinder sets. Then,*

$$\mu\left(\bigcup_{i \in I} \mathbf{P}_i\right) \leq \mu_k\left(\bigcup_{i \in I} \mathbf{P}_i\right).$$

(ii) Let $A \subset \mathbf{P}$, $k \in \mathbb{N}^+$, *and* $I := \left\{i \in M^k : \mathbf{P}_i \cap A\right\} \neq \varnothing$. *Then,*

$$\mu_k(A) \leq \mu\left(\bigcup_{i \in I} \mathbf{P}_i\right).$$

Proof. This lemma was proved in Lemma 8 of [14] for subsets and collections of k-cylinders of the Sierpinski gasket. Notice that for $i = i_1 i_2 i_3 ... i_k \in M^k$, and since $f_{i_k}(z_{i_k}) = z_{i_k}$, the point $f_i(z_{i_k}) = f_{i(k-1)}(z_{i_k})$ belongs to $\mathbf{P}_i \cap A_k$ (recall that $A_k = F^{k-1}(A_1)$, so $f_i(z_j)$ does not necessarily belong to A_k). Therefore, each k-cylinder contains at least a point in A_k. We claim that $\mathbf{P}_i \cap A_k$ is indeed a singleton for any $k \in \mathbb{N}^+$ and $i \in M^k$. This is true for $k = 1$. Let us assume that it is also true for some natural number k, and that there is some $(k+1)$-cylinder $\mathbf{P}_i$ and two points, x and y, both in $A_{k+1} \cap \mathbf{P}_i$. Then, there are indexes j^x and j^y in Σ with $\pi(j^x) = x$ and $\pi(j^y) = y$ and with $(k+1)$-curtailments $i^x(k+1) = j^x(k+1) = i$ (see Sec. 3.1 for a definition of π and the curtailments). Let σ be defined by $\sigma(i) := i_2 i_3 ... i_{k+1}$ for $i \in M^{k+1}$ and $\sigma(i) = i_2 i_3 ....$ for $i \in \Sigma$, and let $\mathcal{T}$ be the geometric shift correspondence (see (3.2)). We know that $j^x \in \pi^{-1}(x)$, so $\pi \circ \sigma(j^x) \in \mathcal{T}(x)$, and $\sigma(j^x(k+1)) = \sigma(i)$, so $\pi \circ \sigma(j^x(k+1)) \in \pi(\sigma(i)) = \mathbf{P}_{\sigma(i)}$, and $\pi \circ \sigma(j^x) \in \mathcal{T}(x) \cap \mathbf{P}_{\sigma(i)}$. Analogously, $\pi \circ \sigma(j^y) \in \mathcal{T}(y) \cap \mathbf{P}_{\sigma(i)}$. We know that, in $\mathbf{P}$, the points for which $\mathcal{T}(x)$ is not a singleton are the five points where two 1-cylinders touch, and they all belong to A_2, and we also know that $\mathcal{T}(A_2) = A_1$. This, together with $A_{k+1} = F(A_k)$, implies that $\mathcal{T}(A_{k+1}) = A_k$. Therefore, both $\pi \circ \sigma(j^x) = f_i^{-1}(x)$ and $\pi \circ \sigma(j^y) = f_i^{-1}(x)$, belong to $\mathbf{P}_{\sigma(i)} \cap A_k$, which gives a contradiction that proves our claim.

Now, the proof of Lemma 6.1 for the Sierpinski gasket, given in Lemma 8 of [14], is based on two properties of the Sierpinski gasket: that any k-cylinder has a unique point in A_k, and that any point in A_k belongs to either a unique k-cylinder or exactly two k-cylinders. These two properties turn out to be true for $\mathbf{P}$ as well. Therefore, the arguments given for the proof of Lemma 8 in [14] also work here, and this completes the proof. □

In Lemma 6.2, the μ-measure of balls $B(x,d)$ is compared to the μ_k-measure of balls centred at points close to x and in A_k. Here, $\overset{\circ}{B}$ denotes the topological interior of $B \subset \mathbb{R}^2$. Note that $\mu(B(x,d)) = \mu(\overset{\circ}{B}(x,d))$, but the equality does not hold for the measures μ_k if some point of A_k belongs to the boundary of $B(x,d)$. The difference between Parts (ii) and (iii) in the lemma below is that in (iii) there is a point of A_k in the boundary of the balls $B(x, d_{\min}(k))$ and $B(x, d_{\max}(k))$. Our algorithms are designed to compute the μ_k-density of balls centred at points of A_k and with some point of A_k in their boundary. In order to give lower and upper bounds for $P^s(\mathbf{P})$ and $C^s(\mathbf{P})$ based on the measures μ_k and the sets A_k, we will use Part (iii) of the lemma below. This is why we will estimate $P^s(\mathbf{P})$ using open balls and $C^s(\mathbf{P})$ using closed balls (see Remark 6.4). Recall that ϕr^k is the diameter of all the k-th cylinder sets.

Lemma 6.2. *Let k be a positive integer.*
(i) If $x \in \mathbf{P}$ and $d > \phi r^k$, then there is a point $y_k \in A_k$ such that

$$\mu(B(y_k, d - \phi r^k)) \leq \mu(B(x,d)) \leq \mu(\overset{\circ}{B}(y_k, d + \phi r^k)).$$

(ii) For any $x \in \mathbb{R}^2$ and any d such that $d - \phi r^k > 0$,

$$\mu_k(B(x, d - \phi r^k)) \leq \mu(B(x,d)) \leq \mu_k(\overset{\circ}{B}(x, d + \phi r^k)).$$

(iii) Given a ball $B(x,d)$, if $\mathbf{P} \nsubseteq B(x,d)$, $d > \phi r^k$ and $B(x,d) \cap A_k \neq \varnothing$, then there are points y_k, $z_k \in A_k$, such that

$$\mu_k(\overset{\circ}{B}(x, d_{\min}(k))) \leq \mu(B(x,d)) \leq \mu_k(B(x, d_{\max}(k))), \tag{6.1}$$

where $d_{\min}(k) := |y_k - x|$, $d_{\max}(k) := |z_k - x|$ and

$$d - \phi r^k \leq d_{\max}(k) \leq d + \phi r^k, \tag{6.2}$$

$$d - \phi r^k \leq d_{\min}(k) \leq d + \phi r^k. \tag{6.3}$$

Proof. (i) Let $i \in M^k$ be such that $x \in \mathbf{P}_i$, and let $y_k \in \mathbf{P}_i \cap A_k$. The inclusions $B(y_k, d - \phi r^k) \subset B(x,d)$ and $\overset{\circ}{B}(x,d) \subset \overset{\circ}{B}(y_k, d + \phi r^k)$ give the first inequality and the second one respectively.
(ii) Let $J_k = \big\{i \in M^k : \mathbf{P}_i \subset B(x,d)\big\}$ and

$$H_k := \big\{i \in M^k : B(x, d - \phi r^k) \cap \mathbf{P}_i \neq \varnothing\big\}.$$

Clearly, $H_k \subset J_k$. Then,

$$\mu_k(B(x, d - \phi r^k)) \leq \mu(\cup_{i \in H_k} \mathbf{P}_i) \leq \mu(\cup_{i \in J_k} \mathbf{P}_i) \leq \mu(B(x,d)),$$

where the first inequality holds by Lemma 6.1 (ii). Now, let

$$I_k := \left\{ i \in M^k : \mathbf{P}_i \cap \overset{\circ}{B}(x,d) \neq \varnothing \right\}.$$

Clearly, $\cup_{i \in I_k} \mathbf{P}_i \subset \overset{\circ}{B}(x, d + \phi r^k)$. This, together with Lemma 6.1 (i), gives

$$\begin{aligned} \mu(B(x,d)) &= \mu(\overset{\circ}{B}(x,d)) \leq \mu(\cup_{i \in I_k} \mathbf{P}_i) \\ &\leq \mu_k(\cup_{i \in I_k} \mathbf{P}_i) \leq \mu_k(\overset{\circ}{B}(x, d + \phi r^k)). \end{aligned}$$

(iii) Notice first that the set of indexes

$$G_k := \{i \in M^k : \partial B(x,d) \cap \mathbf{P}_i \neq \emptyset\}$$

is non empty because, if $G_k = \varnothing$, then $\mathbf{P} \subset \mathring{B}(x,d) \cup \left(\mathring{B}(x,d)\right)^c$, and using that $B(x,d) \cap A_k \neq \varnothing$ and $\mathbf{P} \nsubseteq B(x,d)$, both $\mathring{B}(x,d) \cap \mathbf{P}$ and $\left(\mathring{B}(x,d)\right)^c \cap \mathbf{P}$ would be non empty, which contradicts that $\mathbf{P}$ is a connected set.
Let $I_k := \{i \in M^k : B(x,d) \cap \mathbf{P}_i \neq \emptyset\}$ and

$$d_{\max}(k) := \max\{|y-x| : \ y \in A_k \cap (\cup_{i \in I_k} \mathbf{P}_i)\}.$$

Then, we have

$$\mu(B(x,d)) \leq \mu(\cup_{i \in I_k} \mathbf{P}_i) \leq \mu_k(B(x, d_{\max}(k))). \tag{6.4}$$

For the second inequality in (6.4), notice that by Lemma 6.1 (i),

$$\mu(\cup_{i \in I_k} \mathbf{P}_i) \leq \mu_k(\cup_{i \in I_k} \mathbf{P}_i) = \mu_k((\cup_{i \in I_k} \mathbf{P}_i) \cap A_k) \leq \mu_k(B(x, d_{\max}(k))).$$

This gives the second inequality in (6.1).

Now, let $z_k \in A_k$ be such that $d_{\max}(k) = |z_k - x|$. Obviously, $d_{\max}(k) \leq d + \phi r^k$, and using that $\partial B(x,d) \cap (\cup_{i \in M^k} \mathbf{P}_i) \neq \varnothing$, we see that

$$d + \phi r^k \geq d_{\max}(k) \geq \max\{|y-x| : \ y \in A_k \cap (\cup_{i \in G_k} \mathbf{P}_i)\} \geq d - \phi r^k,$$

and (6.2) is proved. Let $T_k := \{i \in M^k : \mathbf{P}_i \cap B(x,d)^c \neq \emptyset\}$ and

$$d_{\min}(k) := \min\{|x-y| : \ y \in A_k \cap (\cup_{i \in T_k} \mathbf{P}_i)\}.$$

Notice that

$$\mu_k(\mathring{B}(x, d_{\min}(k))) = \mu_k(\mathring{B}(x, d_{\min}(k)) \cap A_k) = \mu_k\left(\left(\cup_{i \in T_k^c} \mathbf{P}_i\right) \cap A_k\right) \tag{6.5}$$

because no point p in A_k with $d(x,p) < d_{\min}(k)$ can belong to some some cylinder $\mathbf{P}_i$ with $i \in T_k$. Now, by Lemma 6.1 (ii), and using that $\cup_{i \in T_k^c} \mathbf{P}_i \subset B(x,d)$, we have

$$\mu_k\left(\left(\cup_{i \in T_k^c} \mathbf{P}_i\right) \cap A_k\right) \leq \mu\left(\cup_{i \in T_k^c} \mathbf{P}_i\right) \leq \mu(B(x,d)),$$

which, together with (6.5), gives the first inequality in (6.1). It only remains for us to prove (6.3). Obviously, $d_{\min}(k) \geq d - \phi r^k$, and using that $\partial B(x,d) \cap (\cup_{i \in M^k} \mathbf{P}_i) \neq \varnothing$, we get that

$$d_{\min}(k) \leq \min\{|y-x| : \ y \in A_k \cap (\cup_{i \in G_k} \mathbf{P}_i)\} \leq d + \phi r^k.$$

$\square$

The following theorem is a discrete version of Theorem 4.1 valid for computational purposes. P_k and C_k are the estimates of $P^s(\mathbf{P})$ and $C^s(\mathbf{P})$ at the iteration k of the algorithms.

Theorem 6.3. *Let $\rho = dist(\mathbf{P}_{01}, \partial\mathcal{D})$, $\delta = dist(\mathbf{P}_{01}, \mathbf{P} - \mathbf{P}_0)$ and $r = \frac{\phi}{2\phi+1}$, where ϕ is the golden number.*
(i) For $k \geq 5$, let $D_k^P = \left[r\rho - 2\phi r^k, \rho\right]$, and

$$P_k = \max\left\{\frac{(2d)^s}{\mu_k(\overset{\circ}{B}(x,d))} : \; x \in A_k \cap \mathbf{P}_{01}, \; d = |x-y| \in D_k^P, \; y \in A_k\right\}. \tag{6.6}$$

Then,

$$P_k^{\inf} \leq P^s(\mathbf{P}) \leq P_k^{\sup}, \tag{6.7}$$

where

$$P_k^{\inf} = \frac{(2d_k)^s}{\mu_k(\overset{\circ}{B}(x_k, d_k + \phi r^k))}, \; P_k^{\sup} = K_k^P P_k,$$

$K_k^P = \left(1 - \frac{2\phi r^{k-1}}{\rho}\right)^{-s}$ *and $\overset{\circ}{B}(x_k, d_k)$ is a ball maximising (6.6).*
(ii) Let $k \geq 4$, $D_k^C = \left[r\delta, \delta + 2\phi r^k\right]$, and

$$C_k = \min\left\{\frac{(2d)^s}{\mu_k(B(x,d))} : \; x \in A_k \cap \mathbf{P}_{02}, \; d = |x-y| \in D_k^C, \; y \in A_k\right\}. \tag{6.8}$$

Then,

$$C_k^{\inf} \leq C^s(\mathbf{P}) \leq C_k^{\sup}, \tag{6.9}$$

where

$$C_k^{\inf} = K_k^C C_k, \; C_k^{\sup} = \frac{(2d_k)^s}{\mu_k(B(x_k, d_k - \phi r^k))},$$

$K_k^C = \left(1 + \frac{2\phi r^{k-1}}{\delta}\right)^{-s}$ *and $B(x_k, d_k)$ is a ball minimising (6.8),*

Proof. (i) Let $B(x,d)$ be such that $P^s(\mathbf{P}) = \frac{(2d)^s}{\mu(B(x,d))}$ with $r\rho \leq d \leq \rho$. Since $k \geq 5$ and $\rho \sim 0.0729$, it is easy to check that $r\rho \geq 2\phi r^k$. Using Lemma 6.2 (i), we know that there is an $y_k \in A_k$ such that

$$\mu(B(x,d)) \geq \mu(B(y_k, d - \phi r^k)) \tag{6.10}$$

and, applying Part (iii) of Lemma 6.2 to the ball $B(y_k, d - \phi r^k)$, we know that there is a point $\widetilde{y}_k \in A_k$ with

$$d - 2\phi r^k \leq d^* := |\widetilde{y}_k - y_k| \leq d \tag{6.11}$$

such that

$$\mu(B(y_k, d - \phi r^k)) \geq \mu_k(\overset{\circ}{B}(y_k, d^*)). \tag{6.12}$$

Using (6.10), (6.12), (6.11), that $\rho \geq d \geq r\rho$ and the definition of P_k, we get

$$P^s(\mathbf{P}) = \frac{(2d)^s}{\mu(B(x,d))} \leq \frac{(2d)^s}{\mu_k(\overset{\circ}{B}(y_k, d^*))} = \left(\frac{d}{d^*}\right)^s \frac{(2d^*)^s}{\mu_k(\overset{\circ}{B}(y_k, d^*))}$$
$$\leq \left(\frac{d}{d - 2\phi r^k}\right)^s \frac{(2d^*)^s}{\mu_k(\overset{\circ}{B}(y_k, d^*))} \leq \left(\frac{d}{d - 2\phi r^k}\right)^s P_k$$
$$= \left(1 - \frac{2\phi r^k}{d}\right)^{-s} P_k \leq \left(1 - \frac{2\phi r^{k-1}}{\rho}\right)^{-s} P_k = P_k^{\sup},$$

where $\mu_k(\overset{\circ}{B}(y_k, d^*)) > 0$ if $d - 2\phi r^k > 0$, which is guaranteed because $k \geq 5$. Using the definition of $P^s(\mathbf{P})$ and Lemma 6.2(ii), we get

$$P^s(\mathbf{P}) = \frac{(2d)^s}{\mu(B(x,d))} \geq \frac{(2d_k)^s}{\mu(B(x_k, d_k))} \geq \frac{(2d_k)^s}{\mu_k(\overset{\circ}{B}(x_k, d_k + \phi r^k))} = P_k^{\inf}.$$

(ii) Let $B(x,d)$ be such that $C^s(\mathbf{P}) = \frac{(2d)^s}{\mu(B(x,d))}$ with $r\delta \leq d \leq \delta$. Since $k \geq 4$, it is easy to check that $r\delta \geq \phi r^k$. By Lemma 6.2 (i), we know that there is $y_k \in A_k$ such that

$$\mu(B(x,d)) \leq \mu(B(y_k, d + \phi r^k)). \tag{6.13}$$

From Part (iii) of Lemma 6.2 applied to the ball $\mu(B(y_k, d + \phi r^k))$, we know there is a $z_k \in A_k$ such that

$$\mu(B(y_k, d + \phi r^k)) \leq \mu_k(B(y_k, d^*)) \tag{6.14}$$

with

$$d \leq d^* := |z_k - y_k| \leq d + 2\phi r^k. \tag{6.15}$$

The inequalities in (6.13), (6.14), (6.15) and that $\delta \geq d \geq r\delta$ together with the definition of C_k, give

$$C^s(\mathbf{P}) = \frac{(2d)^s}{\mu(B(x,d))} \geq \frac{(2d)^s}{\mu_k(B(y_k, d^*))} = \left(\frac{d}{d^*}\right)^s \frac{(2d^*)^s}{\mu_k(B(y_k, d^*))} \geq$$
$$\left(\frac{d}{d + 2\phi r^k}\right)^s \frac{(2d^*)^s}{\mu_k(B(y_k, d^*))} \geq \left(\frac{d}{d + 2\phi r^k}\right)^s C_k =$$
$$\left(1 + \frac{2\phi r^k}{d}\right)^{-s} C_k \geq \left(1 + \frac{2\phi r^k}{r\delta}\right)^{-s} C_k = C_k^{\inf}.$$

$C_k^{\sup}$ is obtained using the definition of $C^s(\mathbf{P})$ and Part (ii) of Lemma 6.2,

$$C^s(\mathbf{P}) = \frac{(2d)^s}{\mu(B(x,d))} \leq \frac{(2d_k)^s}{\mu(B(x_k, d_k))} \leq \frac{(2d_k)^s}{\mu_k(B(x_k, d_k - \phi r^k))} = C_k^{\sup}$$

where $\mu_k(B(x_k, d_k - \phi r^k)) > 0$ because $k \geq 4$. □

Remark 6.4. Notice that we have defined C_k using closed balls. The reason is that our algorithm works with balls centred at points of A_k and with some point of A_k on its boundary. In the proof of the inequality $C^s(\mathbf{P}) \geq C_k^{\inf}$, we have used (6.14), which does not work for the open ball $\overset{\circ}{B}(y_k, d^*)$. For $C_k^{\sup}$, we can use open or closed balls. We have used closed balls for two reasons. The first is that in the definition of C_k we have used closed balls, and in this way we do not change the type of ball that we use in the algorithm. The second is that $\mu_k(B(x_k, d_k - \phi r^k)) \geq \mu_k(\overset{\circ}{B}(x_k, d_k - \phi r^k))$, and therefore, using a closed ball gives an upper bound, $C_k^{\sup}$, for $C^s(\mathbf{P})$ lower or equal than that obtained by taking the open ball. For analogous reasons, we have defined P_k and $P_k^{\inf}$ using open balls.

7. Numerical results

The algorithms for the estimation of $P^s(\mathbf{P})$ and $C^s(\mathbf{P})$ are based on parts (i) and (ii) of Theorem 6.3 respectively. They are similar to those described in [13] and [14] for the estimation of the packing and centred Hausdorff measures of the Sierpinski gasket, respectively. The modifications required for the adaptation to the penta-Sierpinski gasket are obvious using Theorem 6.3.

7.1. Estimates and bounds for $P^s(\mathbf{P})$ and $C^s(\mathbf{P})$

The estimates P^k of $P^s(\mathbf{P})$ and its bounds $P_k^{\inf}$ and $P_k^{\sup}$ for $k \in \{7, ..., 10\}$ are given in Table 1. An optimal ball at iteration k is $\overset{\circ}{B}(x_k, d_k)$, where $d_k = |y_k - x_k|$, $x_k = f_{i(k-1)}(z_{i_k})$, $y_k = f_{j(k-1)}(z_{j_k})$ and $i(k)$, $j(k) \in M^k$ are the codes that correspond to x_k and y_k, respectively. For any k, we obtain that the centre of a ball that maximises the μ_k-inverse density is $x = f_{01}(z_0) \sim (-0.736068, 0.038352)$, that $P^s(\mathbf{P}) \sim P_{10} \sim 6.775$, and that $P^s(\mathbf{P}) \in \left[P_k^{\inf}, P_k^{\sup}\right] \sim [6.728, 6.863]$.

The estimates C_k of $C^s(\mathbf{P})$ and its bounds $C_k^{\inf}$ and $C_k^{\sup}$ for $k \in \{7, ..., 10\}$ are given in Table 2. We get that $C^s(\mathbf{P}) \sim C_{10} \sim 2.440$, and that $C^s(\mathbf{P}) \in \left[C_k^{\inf}, C_k^{\sup}\right] \sim [2.424, 2.445]$.

In Fig. 2, we have plotted two balls that the algorithms give, for $k = 10$, as those of the minimum and maximum μ_k-density, respectively, together with the cylinder set $\mathbf{P}_0$. A zoom that includes only the cylinder sets $\mathbf{P}_{01}$ and $\mathbf{P}_{02}$ is given in Fig. 3. From these balls, an optimal packing and an optimal covering, respectively, can be built through copies that exhaust $\mathbf{P}$, using the self-similar tiling method (see [18]).

7.2. Estimates of the asymptotic spectrum of densities of $\mathbf{P}$

The estimates of $Spec(\alpha, \mathbf{P})$, $\alpha \in \{\mu, C^s\lfloor_{\mathbf{P}}, P^s\lfloor_{\mathbf{P}}\}$, are given in the following Corollary of Theorem 5.1.

Corollary 7.1. *Let* $\mathbf{P}$ *be the penta-Sierpinski gasket. Then,*
(i) $Spec(\mu, \mathbf{P}) \sim [0.1476, 0.4098]$ *and*

$$[0.1487, 0.4090] \subset Spec(\mu, \mathbf{P}) \subset [0.1466, 0.4125]\,.$$

(ii) $Spec(P^s\lfloor_{\mathbf{P}}, \mathbf{P}) \sim [1, 2.7765]$ *and*

$$[1, 2.7520] \subset Spec(P^s\lfloor_{\mathbf{P}}, \mathbf{P}) \subset [1, 2.8124].$$

(iii) $Spec(C^s\lfloor_{\mathbf{P}}, \mathbf{P}) \sim [0.3663, 1]$ *and*

$$[0.3633, 1] \subset Spec(C^s\lfloor_{\mathbf{P}}, \mathbf{P}) \subset [0.3555, 1].$$

Proof. We know (see (3.11) and (5.2)) that

$$Spec(\alpha, \mathbf{P}) = \alpha(\mathbf{P}) Spec(\mu, \mathbf{P}) = \alpha(\mathbf{P}) \left[\frac{1}{P^s(\mathbf{P})}, \frac{1}{C^s(\mathbf{P})} \right], \tag{7.1}$$

for $\alpha \in \mathcal{M}^s\lfloor_{\mathbf{P}}$. Using (7.1), (6.9) and (6.7), we have that

$$Spec(\mu, \mathbf{P}) \sim \left[\frac{1}{P_{10}}, \frac{1}{C_{10}} \right], \left[\frac{1}{P_{10}^{\inf}}, \frac{1}{C_{10}^{\sup}} \right] \subset Spec(\mu, \mathbf{P}) \subset \left[\frac{1}{P_{10}^{\sup}}, \frac{1}{C_{10}^{\inf}} \right],$$

$$Spec(P^s\lfloor_{\mathbf{P}}, \mathbf{P}) \sim \left[1, \frac{P_{10}}{C_{10}} \right], \left[1, \frac{P_{10}^{\inf}}{C_{10}^{\sup}} \right] \subset Spec(P^s\lfloor_{\mathbf{P}}, \mathbf{P}) \subset \left[1, \frac{P_{10}^{\sup}}{C_{10}^{\inf}} \right],$$

and

$$Spec(C^s\lfloor_{\mathbf{P}}, \mathbf{P}) \sim \left[\frac{C_{10}}{P_{10}}, 1 \right], \left[\frac{C_{10}^{\sup}}{P_{10}^{\inf}}, 1 \right] \subset Spec(C^s\lfloor_{\mathbf{P}}, \mathbf{P}) \subset \left[\frac{C_{10}^{\inf}}{P_{10}^{\sup}}, 1 \right].$$

The proof is completed with the estimates of Tables 1 and 2. □

See in Fig. 4 the graph of $\theta^s_{\mu_{10}}(z_0, d)$ as a function of d (recall that z_0 is the fixed point of f_0, see Fig. 1). For any small value of d, the ball $B(z_0, d)$ is μ_{10}-density equivalent to $B(x, d)$, where $x = f_{01}(z_0)$, which is the centre of a ball of minimum μ_{10}-density in the set of admissible balls.

We have also plotted the graph of $\theta^s_{\mu_{10}}(x^*, d)$ as a function of d, where $x^* = f_{0202}(z_3)$ is the centre of a ball of maximum μ_{10}-density. Notice that, for small values of d, the balls $B(x^*, d)$, $B(z_0, d)$ and $B(x, d)$ with $x = f_{01}(z_0)$ are μ-density equivalent. Therefore, from the information given by μ_{10}, we can conjecture that x^* is the center of balls of minimum and maximum μ-density.

8. Conclusions

We have shown that $\mathbf{P}$ is an α-exact self-similar set, for $\alpha \in \{C^s\lfloor_{\mathbf{P}}, P^s\lfloor_{\mathbf{P}}\}$, and we have computed, with explicit error bounds, the values of $P^s(\mathbf{P})$ and $C^s(\mathbf{P})$. The optimal balls selected by our algorithms are plotted in Fig. 2. Observe that the ball selected as optimal for the packing measure (minimum density) contains almost all the subcylinder $\mathbf{P}_{0100}$ and also small parts of the adjacent subcylinders $\mathbf{P}_{0104}$ and $\mathbf{P}_{0101}$. Thus, even if we were able to find some necessary first-order condition derived from the potential *a.e.*-differentiability properties of $\theta^s_\mu(x, d)$ that could give the exact locations of the centres of the optimal balls, the determination of their exact radii, which may depend, for instance, on the amount of the cylinders $\mathbf{P}_{0104}$ and $\mathbf{P}_{0101}$ that must be aggregated to the cylinder $\mathbf{P}_{0100}$, seems essentially to be a computational issue.

The proof of the main theorem of the paper, Theorem 4.1, could be summarised in the following way. First, in the search for an optimal ball $B(x^*, r^*)$, the domain of the centre x^* is reduced to a subcylinder of the second generation, and the domain of the radius r^* is bounded above. This is achieved through similarities of the system Ψ, rotations and symmetries. Second, using expansive homotheties, the domain of the radius is bounded below. In the case of C^s the expansive homotheties have the centres in the overlapping points. It is apparent that this procedure can be applied to the whole family of regular fractal just-touching polygons or m-Sierpinski gaskets, $m \geq 5$, i.e., just-touching self-similar sets generated by families of m homotheties with centres at the vertexes of a regular polygon of m sides. According to this conjecture, any member of this family would be P^{s_m} and C^{s_m} exact (with s_m the similarity dimension) and A-computable, and its spectrum of asymptotic densities would consist of a unique interval, just as in the case of the penta-Sierpinski gasket.

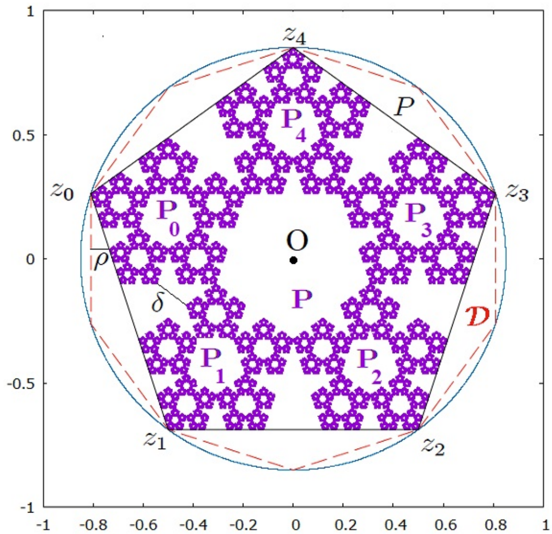

FIGURE 1. Penta-Sierpinski gasket $\mathbf{P}$, the regular pentagon P, the regular decagon $\mathcal{D}$, which is a feasible open set for $\mathbf{P}$, the fixed points, $\{z_i\}_{i\in M}$, of the similitudes in Ψ, the 1-cylinder sets $\mathbf{P}_i$, $i \in M$, and the distances ρ and δ.

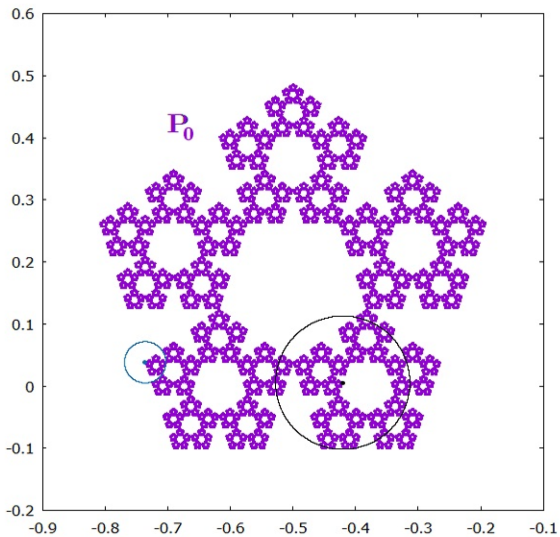

FIGURE 2. Balls of minimum μ_{10}-density (the smaller ball) and maximum μ_{10}-density. The inverses of such densities give the estimates of $P^s(\mathbf{P})$ and $C^s(\mathbf{P})$, respectively.

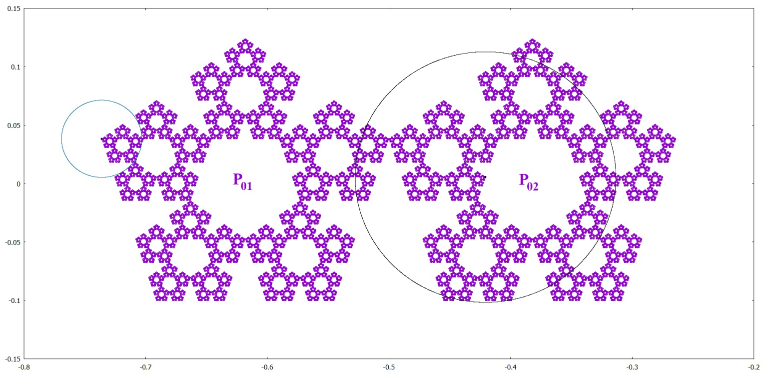

FIGURE 3. Zoom of the balls of minimum and maximum μ_{10}-density that includes only the cylinders $\mathbf{P}_{01}$ and $\mathbf{P}_{02}$.

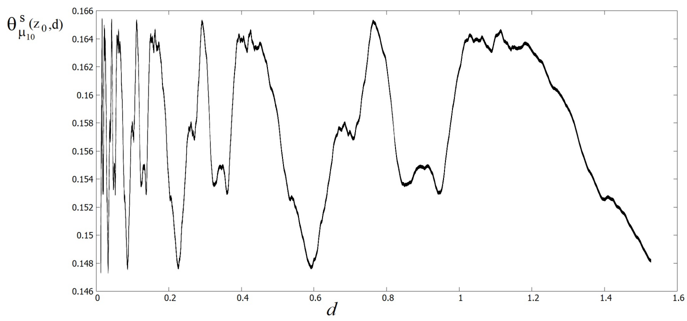

FIGURE 4. μ_{10}-density function of the point z_0.

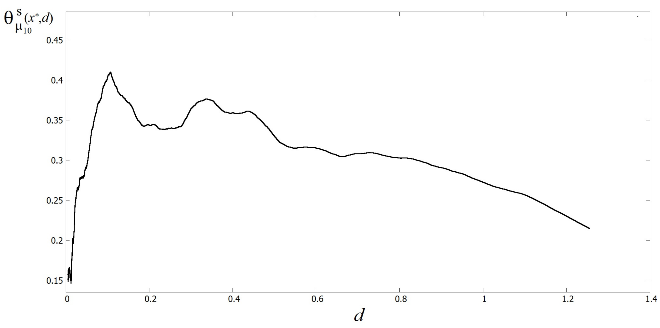

FIGURE 5. μ_{10}-density function of the point x^*, which is a point of maximum μ_{10}-density

	$k=7$	$k=8$	$k=9$	$k=10$
$i(k)$	0100000	01000000	010000000	0100000000
$j(k)$	0101044	01010444	010104444	0100222012
d_k	0.032852	0.032852	0.032852	0.033029
$P_k^{\inf}$	5.838029	6.502070	6.662043	6.728310
P_k	7.035573	6.848270	6.785049	6.774848
$P_k^{\sup}$	9.014741	7.496194	7.019392	6.862731

TABLE 1. Algorithm outputs rounded to six decimal places: to the smallest value for $P_k^{\inf}$, to the largest for $P_k^{\sup}$ and to the nearest for P_k and d_k.

	$k=7$	$k=8$	$k=9$	$k=10$
$i(k)$	0202333	02023332	020233333	0202333333
$j(k)$	0132240	02214333	013314020	0224222120
d_k	0.103767	0.106246	0.107265	0.107201
$C_k^{\inf}$	2.173163	2.336451	2.399232	2.424525
C_k	2.429249	2.440161	2.439691	2.440110
$C_k^{\sup}$	2.554843	2.474785	2.451908	2.444852

TABLE 2. Algorithm outputs rounded to six decimal places: to the smallest value for $C_k^{\inf}$, to the largest for $C_k^{\sup}$ and to the nearest for C_k and d_k.

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